SET THEORY AT CÓRDOBA

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Set Theory is a new research area in Argentina, still with very few practitioners. We present some of the first steps towards its development at the National University of Córdoba. Cantor's *continuum problem*, that of the determining which place does the cardinality of the reals occupy in the cardinal line, provides an appropriate frame for this exposition (and for the whole of Set Theory indeed).

1. A fruitful strategy for the Continuum Problem

The *Continuum Hypothesis* (*CH*) is the statement that $|\mathbb{R}|$ takes the smallest possible value: "Every uncountable $X \subseteq \mathbb{R}$ is in bijective correspondence with $\mathbb{R}$." Cantor proposed the following elegant strategy for showing *CH*:

(1) Consider X in increasing degree of "topological complexity."
(2) Choose a combinatorially versatile set $\mathcal{C}$ such that $\mathbb{R} \xhookrightarrow[\text{onto}]{\text{1-1}} \mathcal{C} \xhookrightarrow{\text{1-1}} X$ can be proved for any X.

The simplest case of the above (apart from the trivial one for open X) is that of nonempty perfect sets P of reals, those satisfying

$$P = P' := \{\text{accumulation points of } P\}. \tag{1.1}$$

Every such P satisfies $|P| = |\mathbb{R}|$, and the proof motivates the choice of $\mathcal{C}$. Define recursively, by using the fact that no point in P is isolated and the fact that P is Hausdorff, nonempty open $A_{\mathbf{b}}$, where $\mathbf{b} = b_1 b_2 \dots b_n$ is a binary sequence, such that

- $\overline{A_{\mathbf{b}0}} \cap \overline{A_{\mathbf{b}1}} = \emptyset$;
- $\overline{A_{\mathbf{b}0}}, \overline{A_{\mathbf{b}1}} \subseteq A_{\mathbf{b}}$;
- $\operatorname{diam} A_{\mathbf{b}} < \frac{1}{2^n}$.

This stipulation induces an injection ι from $\mathcal{C} := \{0,1\}^{\mathbb{N}}$ to P given by

$$\iota(b_1 b_2 \dots) := \text{the unique element of } \bigcap_n A_{b_1 b_2 \dots b_n}$$

which actually is a continuous embedding, when $\mathcal{C}$ is topologized as the countable product of discrete copies of $\{0,1\}$.

The "Cantor scheme" $\{A_{\mathbf{b}}\}_{\mathbf{b}}$ is partially ordered by (reverse) inclusion as a *tree*: the set of predecessors $x{\downarrow}$ of any element x is a chain. It is convenient to consider its skeleton to be the poset $2^{<\omega}$ of all finite binary sequences $\mathbf{b}$ under end-extension. Trees of sequences, and more generally, trees in which $x{\downarrow}$ is always a well-order are a widespread tool in modern Set Theory.

In what might have been the earliest use of an operation involving an infinite set as an argument [14, Sect 1.2] (and definitely the first transfinite process), Cantor iterated the "derivative" operation in (1.1) from an uncountable closed set X

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Supported by Secyt-UNC project 33620230100751CB and Conicet PIP project 11220210100508CO.

obtaining a well-ordered sequence:

$$X \longmapsto X' \longmapsto X'' \longmapsto X''' \longmapsto \cdots \longmapsto X^{(\infty)} \longmapsto \left(X^{(\infty)}\right)' \longmapsto \cdots \tag{1.2}$$

where the "limit" step $X^{(\infty)}$ is defined as the intersection $\bigcap_{n\in\mathbb{N}_0} X^{(n)}$. By using second countability, it can be seen that after countably many steps the sequence (1.1) stabilizes in a perfect set, thus reducing the general case of closed X to the previous one.

Ordinals were defined to be indices of any such "longer-than-infinity" construction, and they can be construed as isomorphism types of well-orders. One fundamentally recurrent theme in Set Theory is the appearance of well-ordered and well-founded hierarchies. As a sample of results, consider the relation $\preceq$ of embeddability between linear orders. We have:

Theorem 1.1 (Laver [16]). *The class of* σ-scattered *linear orders (countable unions of orders not embedding* $\mathbb{Q}$*) is well-founded under* $\preceq$*.*

All countable orders are σ-scattered, so this resolved Fraïssé's Conjecture [4]. For uncountable orders, $\preceq$ is not well-founded but we have the following conditional result:

Theorem 1.2 (Moore [18]). *The* Proper Forcing Axiom *implies that the class of uncountable orders have five* $\preceq$*-minimal elements (modulo bi-embeddability): the first uncountable ordinal, its converse, any uncountable set of reals of minimum cardinality, a* Countryman line*, and its converse.*

The last linear order is a particular case of an *Aronszajn line*. Aronszajn lines are uncountable linear orders that contain no increasing nor decreasing uncountable well-ordered subsets, but are not order-isomorphic to $\mathbb{R}$.

The standard construction of Aronszajn lines is based in turn in Aronszajn trees, which is another example of tree of (ordinal-indexed) sequences. In Argentina, Ricardo Ricabarra pioneered the study of such trees in the late 50s publishing the monograph [22]. Today, Gervasio Figueroa is studying the classical results on Aronszajn and Suslin trees during his BSc. ("Licenciatura") in Mathematics at Córdoba, and their recent applications to General Topology [17].

2. Classification problems

Bernstein, a student of Cantor, showed that the strategy from the previous section could not be used for arbitrary $X \subseteq \mathbb{R}$, by constructing a set B such that neither B nor $\mathbb{R} \setminus B$ contains an nonempty perfect set.

Nevertheless, the first item of the strategy can be interpreted in a useful way these days. Wadge reduction classifies sets under topological complexity: Let X, Y be topological spaces, $A \subseteq X$, and $B \subseteq Y$. A is *Wadge-reducible* to B ("$A \leq_W B$") if there exists a continuous $f : X \to Y$ such that $A = f^{-1}(B)$. An interesting relation to the previous section emerges:

Theorem 2.1 (Wadge-Martin [15, Theorem 21.15]). *The relation* $\leq_W$ *between Borel subsets of second countable zero-dimensional spaces is well-founded.*

Pequignot [21] obtained similar well-foundedness results for an adaptation of $\leq_W$ for Borel subspaces of a Polish space (e.g., Euclidean n-space, $\{0,1\}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$).

Of particular recent relevance is the study of Borel reductions between relations. Let now $R \subseteq X \times X$ and $S \subseteq Y \times Y$ be binary relations; R is *Borel reducible* to S ("$R \leq_B S$") if there is a Borel $f : X \to Y$ such that $a \mathrel{R} b \iff f(a) \mathrel{S} f(b)$, for any $a, b \in X$. A prominent example of this kind of reduction is the case where the space X consists of structures of some kind (for instance, countable graphs) and R is the corresponding notion of isomorphism. If S is the equality on Y, then f is essentially the same as a complete assignment of invariants chosen from Y. Crucially, if one shows that $(\cong) \not\leq_B (=_Y)$, then there is no possible "definable" complete assignment of invariants from Y to structures from X.

Calibrating the complexity of the isomorphism relation on X by determining which pairs Y, S are viable is called the *classification problem for X*. One of the most important recent results on the topic is the following:

Theorem 2.2 (Paolini, Shelah [19]). *The classification problem for countable torsion-free abelian groups has maximum complexity.*

This means, in particular, that this problem is as difficult as the classification of all countable groups modulo isomorphism. This level of complexity is partially witnessed by the fact that $\cong$ (as a subset of X^2) is not a Borel subset.

At the other extreme of the classification spectrum, *smooth* relations are the ones Borel-reducible to $=_{\mathbb{R}}$. A smooth classification problem admits a definable assignment of real numbers (or, equivalently, sequences of reals) as complete invariants. Related to this notion, the relation E_0 of equality-modulo-finite of binary sequences (or equivalently, of subsets of $\mathbb{N}$) is the the least non smooth Borel relation [13].

It is not necessary to restrict oneself to classify structures modulo isomorphism. In Computer Science, a much weaker relation is extremely important for the proper understanding of systems, that of *bisimilarity*, or "equivalence of behavior" [20, 24]. It admits a neat characterization as a relation between *transition systems*: pointed, directed multigraphs S, where edges are represented by binary relations $R_a^S \subseteq S \times S$ (with a ranging on a fixed set L). A *bisimulation* between $\mathbf{S}, s := \langle S, s, \{R_a^S\}_{a\in L}\rangle$ and $\mathbf{T}, t$ is a relation $B \subseteq S \times T$ such that $s \mathrel{B} t$ and

- $s_1 \mathrel{R_a^S} s_2 \;\&\; s_1 \mathrel{B} t_1 \implies$ there exists t_2 such that $t_1 \mathrel{R_a^T} t_2 \;\&\; s_2 \mathrel{B} t_2$.
- $t_1 \mathrel{R_a^T} t_2 \;\&\; s_1 \mathrel{B} t_1 \implies$ there exists s_2 such that $s_1 \mathrel{R_a^S} s_2 \;\&\; s_2 \mathrel{B} t_2$.

We then say that $\mathbf{S}, s$ and $\mathbf{T}, t$ are *bisimilar* if there exists a bisimulation between them.

We have the following result.

Theorem 2.3 (Sánchez Terraf [23]). *Bisimilarity is not Borel on the space of countable transition systems. Hence it is not smooth.*

The second sentence follows since reductions preserve Borelness. Martín Moroni explored this subject in his 2022 PhD thesis, and we obtained the same non-smoothness result (by a reduction using E_0) for a much restricted class of transition systems, that of well-founded trees of rank $\leq \omega + 2$.

3. Formalization of independence of *CH*

The prospect of a proof of *CH* (or of its negation) were shattered by the groundbreaking work by Kurt Gödel [5] and Paul Cohen [3]: If the current foundations of mathematics are not contradictory, then *CH* cannot be proved nor refuted.

The consistency of the negation of *CH* was obtained by the method of *forcing* devised by Cohen. Forcing soon took the role of the master tool of any set-theorist, showing that many other results where independent from the accepted axioms of Set Theory (in which all of mathematics can be based); these are known by the names of Zermelo, Fraenkel, with a mention of the Axiom of Choice (*ZFC*):

Pairing: For any x, y there exists $\{x, y\}$.
Union: For any x there exists $\bigcup x$.
Infinity: There exists $\omega = \mathbb{N}_0$.
Power Set: For any x there exists $\mathcal{P}(x)$.
Separation: For any x and any definable Q there exists $\{z \in x \mid Q(z)\}$.
Replacement: For any x and any definable F there exists $\{F(z) \mid z \in x\}$.

Choice: (AC) There exists $f : A \to \bigcup A$ such that $\emptyset \neq x \in A$ implies $f(x) \in x$.

Foundation: $\in$ is well-founded.

In a joint work involving Emmanuel Gunther (2019–2020 postdoc), M. Pagano, and M. Steinberg, we devised a computer verification of Cohen's result. To contextualize this achievement, it is helpful to introduce the difference between *computer-assisted* and *computer-formalized* proofs.

Two paradigmatic examples of the first case are the initial proofs of the Four-Color Theorem [1] and Kepler's Conjecture [11]. The first proof was in dispute for several years, and while the second was accepted in the *Annals of Mathematics*, the acceptance letter stated that the referees "[were un]able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem." [12]. In both cases, the difficulty arose because a substantial part of the justification depended on computer calculations. Both the connection of these calculations to the rest of the proof and the correctness of the code itself are highly susceptible to errors and difficult to evaluate. However, the development of more sophisticated computational tools—*proof assistants*—has allowed for the development of proofs that go from the very axioms to the desired result, without omitting any steps whatsoever. There are no lemmas left "to the reader." Although this type of proof may seem impossible in practice, the two quoted theorems have been formalized in this way, respectively by Gonthier [6] and Hales et al. [10].

The formal verification of a result not only provides an extra degree of confidence in it (in fact, there is no dispute regarding Cohen's results) but the required depth of analysis typically allows for obtaining more information from the proof. As an example, during the formalization of Kepler's Conjecture, two open problems were solved, including the Fejes Tóth contact conjecture [10]. Our formalization (involving the development of 34k lines of code, spanning 710+ PDF pages of proofs), allowed us to identify 22 instances of the Replacement Axiom that are sufficient to construct models of *ZFC* satisfying either *CH* or its negation.

The whole formal development is now part of the *Archive of Formal Proofs* of Isabelle [9, 8] and it is presented in the forthcoming [7]. Mateo Marengo Cano is expected to work on the expansion of this formalization during his BSc. in Mathematics.

4. The unexpected appearance of set-theoretic issues

A *hyperplane arrangement* $\mathcal{H}$ is a finite collection of affine subspaces H of $\mathbb{R}^n$ having codimension 1. Any such arrangement determines a partition of the ambient space into (relatively open) *faces*: Each face is one of the possible intersections

$$F := \bigcap_{H \in \mathcal{H}} H^{\epsilon},$$

where H^{ϵ} is either H or one of the two open halves of $\mathbb{R}^n$ determined by H.

Faces are naturally order by inclusion of their closures: $F \leq G \iff \overline{F} \subseteq \overline{G}$. There is also a natural *face semigroup* [2] structure defined on them: $F \cdot G$ is defined as the face neighboring the start of a "generic" open line segment from F to G. The definition ensures the following equivalence:

$$F \cdot G = G \iff F \leq G. \tag{4.1}$$

This product is obviously idempotent and associative (a *band* operation, in the semigroup literature) and maximal elements form a two-sided ideal. Note that

(4.1) can be compared to the relation between $\cup$ and $\subseteq$:

$$F \cup G = G \iff F \subseteq G.$$

Although face semigroups are not commutative, they satisfy the following equation (they are *left regular* bands or "LRB"):

$$F \cdot G \cdot F = F \cdot G. \tag{4.2}$$

It is therefore of interest to characterize which posets $\langle \mathcal{F}, \leq \rangle$ admit an idempotent semigroup operation satisfying (4.1) and (4.2). We call them *associative posets*.

If one strengthens the requirements by asking for commutativity, the answer is well-known: The class of such posets are exactly those in which there exists $\sup\{F, G\}$ for any two $F, G \in \mathcal{F}$, and in this circumstance the product is uniquely determined. This is not the case for the general problem.

Indeed, the characterization of associative posets is not trivial, as the following result shows.

Theorem 4.1 (Petrovich, Sánchez Terraf)**.** *The following are equivalent over ZF:*

(1) Every tree with three levels is associative.
(2) The Axiom of Choice.

Joel Kuperman (current PhD student) extended this result to show that under AC, every *foliated* tree (i.e. in which every element is below some maximal one) is associative. He presented this result among others in the past BLAST 2021 conference, which gathers researchers in Boolean Algebra, Lattice Theory and Set Theory.

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