Permutable bisimulation equivalences of Kripke frames

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Joint work with M. Campercholi and D. Penazzi

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Contents

Motto: Implicit definability $\iff$ Explicit definability.

Example

$\Gamma(p_1, p_2, r) \doteq \{r \to p_1, r \to p_2, p_1 \to (p_2 \to r)\}$.[2]

$$\Gamma(p_1, p_2, r), \Gamma(p_1, p_2, r') \models_{\mathrm{CPC}} r \leftrightarrow r'.$$

Theorem (Beth's for CPC)

Every such definition over CPC *can be made explicit.*

Γ is an implicit definition of $\wedge$ over CPC.
And so is on $\mathrm{CPC}^{\to}$, but no explicit definition here.

[2] Hoogland (2001)

Beth's Theorem

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Frames & BAOs

We will be interested on the (modal) logic of a finite family of finite Kripke frames.

Duality

$$
\begin{array}{rcl}
\text{Kripke Frames} & \longleftrightarrow & \text{Boolean algebras with operators} \\
\mathbf{L} \doteq \langle L, R\rangle & \longleftrightarrow & \mathbf{L}^{\bullet} \doteq \langle \mathcal{P}(L), \cap, \cup, \cdot^{c}, \top, \bot, \Diamond_R \rangle \\
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Equational definition of functions

A function $f : A^n \to A$ is algebraic[3] in $\mathbf{A}$ if there are terms p_i, q_i such that

$$f(\bar{x}) = z \iff \mathbf{A} \models \bigwedge_i p_i(\bar{x}, z) = q_i(\bar{x}, z).$$

Algebraic functions in $\mathbf{A}$ are in correspondence with $\forall\exists! \bigwedge p = q$ sentences holding in $\mathbf{A}$.

Observation

Every new algebraic function gives a counterexample to Beth's Theorem.

[3] Campercholi and Vaggione (2011a).

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Failure of Beth's	$\longleftarrow$	New algebraic function

Characterization of finite $\forall\exists! \bigwedge p = q$ classes

Everything in this slide is true up to iso.

Theorem (Campercholi and Vaggione (2011b))

Let $\mathcal{C}$ a finite class of finite structures. TFAE:

1. *$\mathcal{C}$ is definable by a set of $\forall\exists! \bigwedge p = q$ sentences;*
2. *$\mathcal{C}$ is closed under*
   - *intersection of subalgebras ($\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{C}$ & $\mathbf{B}, \mathbf{C} \leq \mathbf{A} \implies \mathbf{B} \cap \mathbf{C} \in \mathcal{C}$), and*
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For infinite $\mathcal{C}$ is extremely difficult.

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Bisimulation equivalences	$\longleftrightarrow$	Subalgebras

Bisimulation

Definition (Bisimulation)

A relation θ such that whenever $s_1 \; \theta \; t_1$,

- forth if $s_1 \; R \; s_2$ then there is t_2 such that $t_1 \; R \; t_2$ and $s_2 \; \theta \; t_2$.
- back if $t_1 \; R \; t_2$ then there is s_2 such that $s_1 \; R \; s_2$ and $s_2 \; \theta \; t_2$.

Congruence

It's a reflexive, symmetric, and transitive bisimulation.

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Bisimulation equivalences	$\longleftrightarrow$	Subalgebras
Join	$\longleftrightarrow$	Intersection of subalgebras
FP congruences	$\longleftrightarrow$	FP subalgebras

Definition

A ruler is a symmetric reflexive linear Kripke frame:

$$\mathbf{L_n} = \langle n+1, R \rangle, \text{ where } x\, R\, y \iff |x-y| \leq 1.$$

For $n = 5$:

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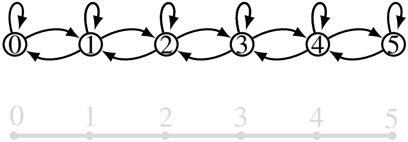

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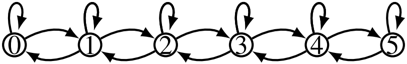
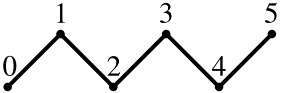


A folding of a ruler . . .

$\langle 1 \rangle$ on $\mathbf{L_5}$



We also allow 1-unit “rests” (as in long staircases): $\langle 2;2 \rangle$ on $\mathbf{L_5}$.

1. Every congruence $\theta \neq \mathbf{L} \times \mathbf{L}$ is a folding and viceversa.
2. θ is trivial $\iff$ $\exists x : x \, \theta \, x+1 \, \theta \, x+2$.

Foldings

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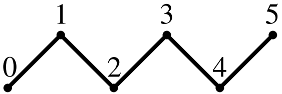

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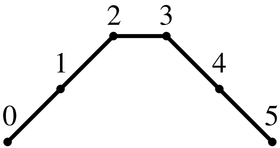

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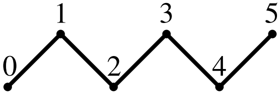

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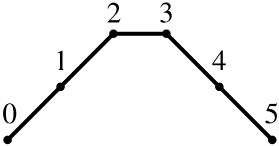

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Two congruences on $\mathbf{L_{18}}$:

$$\theta \vee \delta \quad = \quad \langle k;\bar{r}\rangle \vee \langle l\rangle \qquad k < l$$

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$$\theta \vee \delta = \langle k; \bar{r} \rangle \vee \langle l \rangle \qquad k < l$$

$$\theta \vee \delta = \langle \overset{k}{4}; \overbrace{4, 13}^{\bar{r}} \rangle \vee \langle \overset{l}{6} \rangle$$

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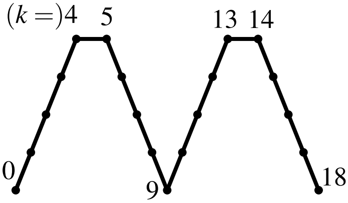
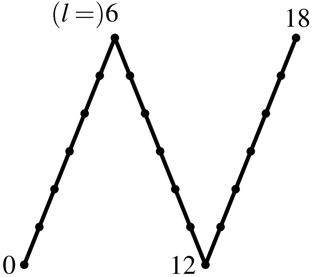
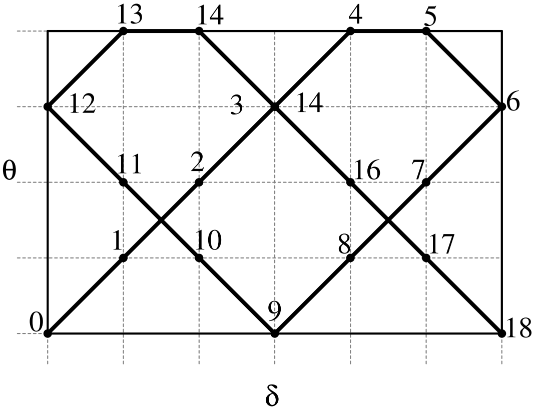


$\vee$



$= ???$

The Join



Some lemmas on trajectories (I)

Lemma

If the trajectory diagram of $\theta \vee \delta$ has two crossings with one coordinate differing in $\frac{1}{2}$, the join is trivial.

Proof.

$$a'' \, \delta \, d \, \theta \, a \, \theta \, c \, \delta \, b \, \theta \, a'.$$

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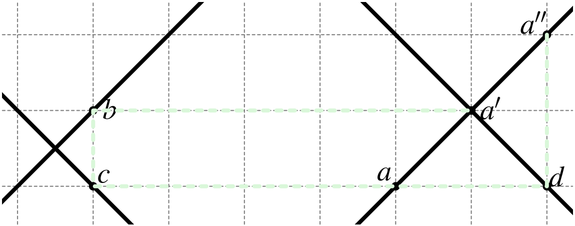

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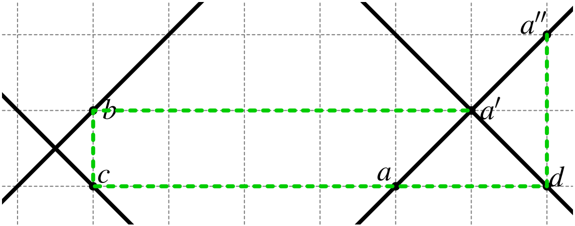

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Lemma

Bounces strictly less than k apart must be of the same type.

Proof.

$a\,\theta\,b\,\delta\,c\,\theta\,d\,\delta\,e$

$a\,\delta\,x\,\theta\,y\,\delta\,z$

Some lemmas on trajectories (II)

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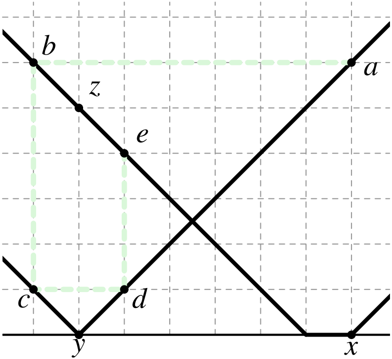

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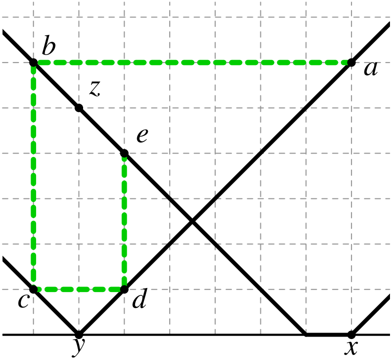

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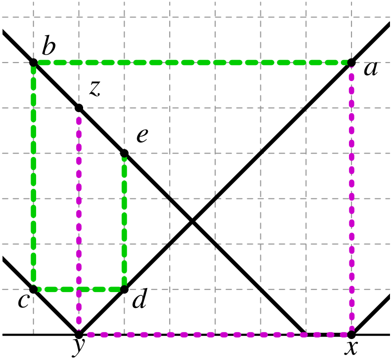

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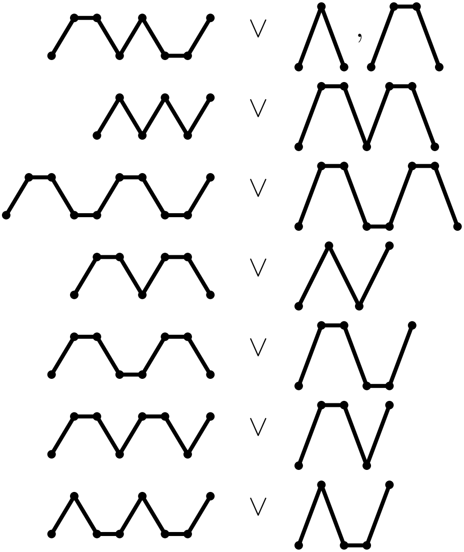
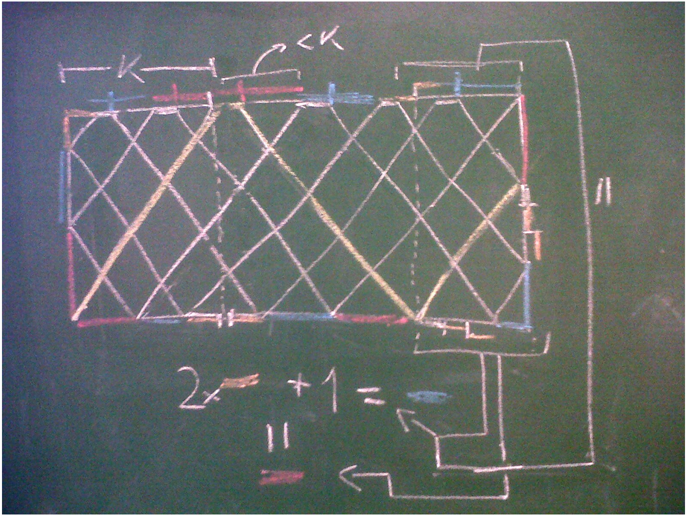
Local non-trivial joins



¡Thank You!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