Advances in Modal Logic, Volume 3

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From Description to Hybrid Logics, and Back

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ABSTRACT. Building on work by Schild, De Giacomo and Lenzerini, we establish a tight connection between description logics and hybrid logics. The main aim of the paper is to provide a modal perspective on some of the distinguishing features of description logic. In particular, by working in a hybrid logic setting we are able to develop a model-theoretic understanding of both assertional and terminological information. We also show how to use the connection between description and hybrid logics to transfer results on complexity and expressive power from one to the other.

1 Introduction

Nearly a decade ago, Schild (1991) observed and exploited the close correspondence between description logics and modal languages. He used it as a bridge to transfer complexity results and axiomatizations from modal logics to description logics, but noticed that the correspondence can only be established at the level of *concept satisfiability*. Basic modal logic is not expressive enough to account for either A-Box reasoning or inference in the presence of definitions (non-empty T-Boxes). Also, some very expressive description languages include constructions for building complex roles such as intersection, converse, and even transitive closure. By lifting the correspondence to Converse Propositional Dynamic Logic (CPDL, Fischer and Ladner 1979), Schild accounted for these constructions and for inference from non-empty T-Boxes. De Giacomo and Lenzerini (1994) extended these results by encoding A-Box reasoning in CPDL. While embeddings of description logics into CPDL have proved useful, they have two important disadvantages. Complexity-wise, the lo-

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Constructor	Syntax	Semantics
concept name	C	$C^{\mathcal{I}}$
top	$\top$	$\Delta^{\mathcal{I}}$
negation ($\mathcal{C}$)	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction ($\mathcal{U}$)	$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
universal quant.	$\forall R.C$	$\{d_1 \mid \forall d_2 \in \Delta^{\mathcal{I}}.(R^{\mathcal{I}}(d_1, d_2) \rightarrow d_2 \in C^{\mathcal{I}})\}$
existential quant. ($\mathcal{E}$)	$\exists R.C$	$\{d_1 \mid \exists d_2 \in \Delta^{\mathcal{I}}.(R^{\mathcal{I}}(d_1, d_2) \wedge d_2 \in C^{\mathcal{I}})\}$
one-of ($\mathcal{O}$)	$\{a_1, \ldots, a_n\}$	$\{d \mid d = a_i^{\mathcal{I}}$ for some $a_i\}$
role filler ($\mathcal{B}$)	$\exists R.\{a\}$	$\{d \mid R^{\mathcal{I}}(d, a^{\mathcal{I}})\}$
role name	R	$R^{\mathcal{I}}$
inverse roles ($\mathcal{I}$)	R^{-1}	$\{(d_1, d_2) \mid R^{\mathcal{I}}(d_2, d_1)\}$

TABLE 1 Common operators of description logics.

cal satisfiability problem of CPDL (i.e., the problem of finding, given a CPDL formula ϕ, a model $\mathcal{M}$ and a state m such that $\mathcal{M}, m \Vdash \phi$) is already EXPTIME-complete, and this prohibits sharp complexity results. And perhaps more crucially if our main aim is to understand the general behavior of description logics, the model theory of CPDL is complex, because the Kleene star (and hence a weak notion of induction) needs to be taken into consideration.

In this paper, we replace CPDL by hybrid languages and in this way shed new light on the issues above. Our main aim is to establish a very tight connection between description and hybrid logics, and to show how it provides a modal perspective on description logics, by means of a number of transparent model-theoretic ideas. We also indicate some possible uses of this connection in terms of results on complexity, expressive power, and meta-logical properties like interpolation and Beth definability.

We start by providing some background. We then recall relevant work by Schild, and De Giacomo and Lenzerini. After that we set up the link between hybrid and description logics, and exploit it.

1.1 Description Logic

Description logics (DLs) are a family of formal languages with a clearly specified semantics, usually in terms of first-order models, together with inference mechanisms to account for knowledge classification. One of the main aims of the field is to identify fragments of first-order logic that are able to capture the features needed for representing a particular problem domain, and which still admit efficient reasoning algorithms.

Let $\mathsf{CON} = \{C_1, C_2, \ldots\}$ be a countable set of *atomic concepts*, $\mathsf{ROL} = \{R_1, R_2, \ldots\}$ a countable set of *atomic roles*, and $\mathsf{IND} = \{a_1, a_2, \ldots\}$ a countable set of *individuals*. For CON, ROL, IND, all pairwise disjoint, $\mathcal{S} = \langle \mathsf{CON}, \mathsf{ROL}, \mathsf{IND} \rangle$ is a *signature*. An *interpretation* $\mathcal{I}$ for $\mathcal{S}$ is a tuple $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set, and $\cdot^{\mathcal{I}}$ assigns elements $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ to constants a_i, subsets $C_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ to atomic concepts C_i, and relations $R_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to atomic roles R_i. The atomic symbols in a DL signature can be combined by means of *concept* and *role constructors*, to form complex expressions. Table 1 defines the constructors for the DLs we will discuss, together with their semantics. It is customary to define systems by postfixing the names of some basic description languages like $\mathcal{AL}$ or $\mathcal{FL}$ with the names of the added operators from Table 1. In this paper, we will be interested in languages having full Boolean expressivity and hence focus on $\mathcal{ALC}$ and its extensions. The most expressive language we will deal with is $\mathcal{ALCOI}$, that is $\mathcal{ALC}$ extended with the "one-of" operator and converse roles (this language can trivially encode the "role-filler" operator $\mathcal{B}$).

In DLs we want to perform inferences given certain background knowledge. Let $\mathcal{L}$ be any description logic, a *knowledge base* Σ in $\mathcal{L}$ is a pair $\Sigma = \langle T, A \rangle$ such that T is the T(erminological)-Box: a finite, possibly empty, set of expressions of the form $C_1 \sqsubseteq C_2$, where C_1, $C_2 \in \mathsf{CON}(\mathcal{L})$ ($C_1 \doteq C_2$ is short for '$C_1 \sqsubseteq C_2$ and $C_2 \sqsubseteq C_1$'). Formulas in T are called *terminological axioms*. In addition, A is the A(ssertional)-Box: a finite, possibly empty, set of expressions of the forms $a : C$ or $(a, b){:}R$ where C is in $\mathsf{CON}(\mathcal{L})$, R is in $\mathsf{ROL}(\mathcal{L})$ and a, b are individuals. Formulas in A are called *assertions*.

Let $\mathcal{I}$ be an interpretation and ϕ a terminological axiom or assertion. Then $\mathcal{I}$ *models* ϕ (notation: $\mathcal{I} \models \phi$) if $\phi = C_1 \sqsubseteq C_2$ and $C_1^{\mathcal{I}} \subseteq {C_2}^{\mathcal{I}}$, or $\phi = a{:}C$ and $a^{\mathcal{I}} \in C^{\mathcal{I}}$, or $\phi = (a, b){:}R$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$. If $\Sigma = \langle T, A \rangle$ is a knowledge base and $\mathcal{I}$ an interpretation, then $\mathcal{I}$ models Σ (notation: $\mathcal{I} \models \Sigma$) if for all $\phi \in T \cup A, \mathcal{I} \models \phi$. Given a knowledge base Σ and a terminological axiom or assertion ϕ, we write $\Sigma \models \phi$ if for all models $\mathcal{I}$ of Σ we have $\mathcal{I} \models \phi$. All standard description logic reasoning tasks (like subsumption or instance checking) can be defined in terms of this relation.

1.2 Hybrid Logic

Modal formulas are evaluated at a given *state* in a model, and their truth values depend on the value of formulas at some related *states*. Yet, nothing in modal syntax gets to grips with the *states* themselves. Hybrid languages are modal languages which solve this "reference problem" by introducing special symbols, called *nominals*, to explicitly name the states in a model.

The basic hybrid language is $\mathcal{H}_{\mathsf{N}}$, basic modal logic extended with nominals. Further extensions are named by listing the added operators.

The most expressive system we will discuss is $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, \mathsf{E}, @)$, the basic hybrid language extended with the converse and existential modalities, and the @ operator. More precisely, let $\mathsf{REL} = \{R_1, R_2, \ldots\}$ be a countable set of *relation symbols*, $\mathsf{PROP} = \{p_1, p_2, \ldots\}$ a countable set of *proposition letters*, and $\mathsf{NOM} = \{i_1, i_2, \ldots\}$ a countable set of *nominals*. $\mathsf{ATOM} = \mathsf{PROP} \cup \mathsf{NOM}$ is the set of *atoms*. The formulas of the hybrid language $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, \mathsf{E}, @)$ in the signature $\langle \mathsf{REL}, \mathsf{PROP}, \mathsf{NOM}\rangle$ are

$$\mathsf{FORMS} := \top \mid a \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \langle R\rangle\phi \mid \langle R^{-1}\rangle\phi \mid \mathsf{E}\phi \mid @_i\phi,$$

where $a \in \mathsf{ATOM}$, $R \in \mathsf{REL}$, $i \in \mathsf{NOM}$, and ϕ, ϕ_1, $\phi_2 \in \mathsf{FORMS}$.

A *hybrid model* $\mathcal{M}$ is a triple $\mathcal{M} = \langle M, \{R_i\}, V\rangle$ where M is a non-empty set, $\{R_i\}$ is a set of binary relations on M, and $V : \mathsf{PROP} \cup \mathsf{NOM} \to \mathit{Pow}(M)$ is such that for all nominals $i \in \mathsf{NOM}$, $V(i)$ is a singleton subset of M. Let $\mathcal{M} = \langle M, \{R_i\}, V\rangle$ be a model and $m \in M$. The interesting cases of the *satisfiability relation* are as follows: $\mathcal{M}, m \Vdash a$ iff $m \in V(a)$, $a \in \mathsf{ATOM}$; $\mathcal{M}, m \Vdash \langle R\rangle\phi$ iff $\exists m' \, (R(m, m')$ & $\mathcal{M}, m' \Vdash \phi)$; $\mathcal{M}, m \Vdash \langle R^{-1}\rangle\phi$ iff $\exists m' \, (R(m', m)$ & $\mathcal{M}, m' \Vdash \phi)$; $\mathcal{M}, m \Vdash \mathsf{E}\phi$ iff $\exists m' \, (\mathcal{M}, m' \Vdash \phi)$; and $\mathcal{M}, m \Vdash @_i\phi$ iff $\mathcal{M}, m' \Vdash \phi$, where $V(i) = \{m'\}$, $i \in \mathsf{NOM}$.

We write $\mathcal{M} \Vdash \phi$ iff for all $m \in M$, $\mathcal{M}, m \Vdash \phi$. This notion extends to sets of formulas in the standard way. A formula ϕ is *satisfiable* if there is a model $\mathcal{M}$ and a state $m \in M$ with $\mathcal{M}, m \Vdash \phi$. A formula ϕ is *valid* if for all models $\mathcal{M}$, $\mathcal{M} \Vdash \phi$. ϕ is a *local consequence* of a set of formulas T (notation: $T \models^{loc} \phi$), if for all models $\mathcal{M}$ and points $m \in M$, $\mathcal{M}, m \Vdash T$ implies $\mathcal{M}, m \Vdash \phi$; ϕ is a *global consequence* of a set of formulas T (notation: $T \models^{glo} \phi$), if for all models $\mathcal{M}$, $\mathcal{M} \Vdash T$ implies $\mathcal{M} \Vdash \phi$. When T is the empty set, we have $\{\} \models^{glo} \phi$ iff $\{\} \models^{loc} \phi$, and simply write $\models \phi$.

2 Schild's Terminologies

It is easy to map concepts in $\mathcal{ALC}$ into formulas of CPDL, while preserving satisfiability — actually, basic poly-modal logic is enough. Just define the translation $\cdot^t$ by putting $(C_i)^t = p_i$, for C_i an atomic concept; $(\neg C)^t = \neg(C^t)$; $(C \sqcap D)^t = C^t \wedge D^t$; and $(\exists R.C)^t = \langle R\rangle C^t$. It is clear that $\cdot^t$ preserves satisfiability. But we need further expressive power to account for T-Box and A-Box reasoning.

Consider the signature $\mathcal{S} = \langle \{C_1, C_2\}, \{R\}, \{a\}\rangle$ and the interpretations $\mathcal{I}_1 = \langle \{m_1, m_2\}, \cdot^{\mathcal{I}_1}\rangle$ and $\mathcal{I}_2 = \langle \{m_3, m_4, m_5\}, \cdot^{\mathcal{I}_2}\rangle$ where $C_1^{\mathcal{I}_1} = \{m_1\}$, $C_2^{\mathcal{I}_1} = \{m_1, m_2\}$, $R^{\mathcal{I}_1} = \{\}$, $a^{\mathcal{I}_1} = m_1$; and $C_1^{\mathcal{I}_2}\{m_4\}$, $C_2^{\mathcal{I}_2} = \{m_3\}$, $R^{\mathcal{I}_2} = \{\}$, $a^{\mathcal{I}_2} = m_5$. Clearly, $\mathcal{I}_1$ models both $C_1 \sqsubseteq C_2$ and $a : C_1$ while $\mathcal{I}_2$ models neither. On the other hand, when we consider $\mathcal{I}_1$ and $\mathcal{I}_2$ as modal models, the relation $\{(m_2, m_3)\}$ is a bisimulation, showing that there are no basic modal formula capturing these notions.

Observe that terminological axioms such as $C_1 \sqsubseteq C_2$ and $a : C_1$ express global properties: they are true in a point in a model iff they are true in all elements of the model. Schild (1991) accounts for the global nature of terminological axioms by using the collapsed model property of CPDL (any satisfiable CPDL formula is satisfiable in a connected model) and the availability of the Kleene star. Due to the former, we can ignore states which are not reachable by a finite sequence of backwards and forwards transitions. And due to the Kleene star we can "compress" these transitions into a single step. Formally, extend $\cdot^t$ by putting $(C \sqsubseteq D)^t = (C^t \rightarrow D^t)$. And for a finite set of terminological axioms T, let T^t be $\bigwedge \phi_i^t$, where $\phi_i \in T$. Now, let $T \cup \{\phi\}$ be a finite set of terminological axioms, and let $R_1, \ldots, R_n$ be all the roles in $T \cup \{\phi\}$. Then $\langle T, \{\}\rangle \models \phi$ iff $\models [(R_1 \cup R_1^{-1} \cup \cdots \cup R_n \cup R_n^{-1})^*]T^t \rightarrow \phi^t$.

As Schild remarks, this translation would not work for an infinite T. On the one hand, T might contain an infinite number of roles, but even in the case of a finite signature, PDL is not compact (Harel 1984, Theorem 2.15), hence inference from infinite sets does not coincide with inference from finite sets. More importantly, lack of compactness has a striking effect on the complexity of the consequence problem, which becomes highly undecidable, an indication that PDL is not computationally well behaved. The computational problems caused by the Kleene star have been well investigated both in the modal and description logic community (Ladner 1977, Halpern and Moses 1992, Sattler 1996, Horrocks and Gough 1997); and authors like Sattler, and Horrocks and Gough have argued that the ability to define a role as transitive often suffices in applications, instead of the full power of transitive closure.

What is going on here? How can we understand terminological axioms in modal terms, using the kind of model-theoretic tools that have been used to analyze modal logic itself? Before we answer this question we highlight one more distinguishing feature of DLs.

3 De Giacomo's Individuals

Accounting for assertional information in CPDL is more complicated than encoding terminological axioms. Below we present a simplified version of a translation proposed by De Giacomo and Lenzerini (1994); the latter enforce the unique name assumption and also deal with complex structure on roles (union, composition, transitive closure, etc.) which makes for the additional complexity. Here, we only discuss the handling of individuals.

Extend the translation $\cdot^t$ defined in Section 2 to assertions by defining

$(a:C)^t = p_a \to C^t$, and $((a,b):R)^t = p_a \to \langle R\rangle p_b$, where p_a and p_b are propositional symbols. Let A be a finite set of assertions, define A^t as $\bigwedge \phi_i^t$ for $\phi_i \in A$. The problem now is that in translating *individuals* as *propositions* in CPDL we have lost the information that individuals denote a single element in the domain. Hence, we have to explicitly force these symbols to behave as individuals. Let $\Sigma = \langle T, A\rangle$ be a knowledge base, $R_1,\ldots,R_n$ the roles appearing in Σ, $a_1,\ldots,a_m$ the individuals mentioned in Σ, and let $\mathsf{SF}(\phi)$ be the set of all subformulas of ϕ. Let $[U]$ stand for $[(R_1 \cup R_1^{-1} \cup \cdots \cup R_n \cup R_n^{-1})^*]$, and let S be a role not in Σ. Let Σ^t be

$$[S][U](A^t \wedge T^t) \wedge$$
$$\bigwedge_{1\leq i\leq m}\left(\langle S\rangle p_{a_i} \wedge \left(\bigwedge_{\psi\in\mathsf{SF}(T^t\wedge A^t)} [S](\langle U\rangle(p_{a_i}\wedge\psi) \to [U](p_{a_i}\to\psi))\right)\right).$$

It can be shown that Σ is consistent if and only if Σ^t is satisfiable. This is enough because in sufficiently expressive languages like the ones we consider in this paper, all standard reasoning tasks can be reduced to knowledge base consistency.

As remarked by Horrocks et al. (2000), De Giacomo's translation is probably too involved and costly to provide effective decision methods. It is also difficult to extract theoretical results from it, except for the general complexity results presented in De Giacomo and Lenzerini (1994). As we already remarked, the model theory of CPDL is intricate because of the inductive nature of the Kleene star, and the cryptic translation provides little help on simplifying things out.

4 Universal Statements and Individuals

The two features of DLs that most modal logicians would probably find hard to understand in terms of the model-theoretic notions they are used to (bisimulations, axiomatizations, etc.), are the use of global information and of information about individuals.

One way to understand the issue of global information is by paying special attention to the standard notions of consequence used in description and modal logics. In Section 1 we introduced two notions of consequence, a *local* one and a *global* one which differ in the way we quantify over states in the model. To understand what is going on, it is helpful to discuss consequence in first-order terms. Given a set of formulas $\Gamma \cup \{\phi\}$ which might contain free variables, the way we define the quantification over models $\mathcal{M}$ and first-order assignments g becomes meaningful. We can either require

(Global) $\forall \mathcal{M}(\forall g \mathcal{M} \models \Gamma[g] \Rightarrow \forall g.\mathcal{M} \models \phi[g])$, or

(Local) $\forall \mathcal{M} \forall g(\mathcal{M} \models \Gamma[g] \Rightarrow \mathcal{M} \models \phi[g])$

The global consequence relation is the one familiar from first-order logic, but first-order consequence is usually analyzed over sets of *sentences*, and in this case both the global and the local notions coincide. Modal languages are usually equivalent to first-order formulas with *free variables*, and choosing one of the two possibilities becomes an issue. In line with the general local perspective of modal logics (evaluation of a formula *at a state* in the model), the local notion of consequence is the most natural. But the presence of the E modality in hybrid languages makes things simpler, as we can easily interdefine local and global consequence. Let A (the universal modality) be the dual of $\Diamond$ (i.e., $\mathsf{A}\phi := \neg\mathsf{E}\neg\phi$), then

$$\Gamma \models^{glo} \phi \text{ iff } \{\mathsf{A}\psi \mid \psi \in \Gamma\} \models^{loc} \phi. \qquad (1)$$

Goranko and Passy (1992) study properties of languages containing E, and prove that global properties of a logic $\mathcal{L}$ correspond to local properties of the logic $\mathcal{L}^{\mathsf{E}}$ which arises from $\mathcal{L}$ by adding E. In particular, for basic modal logics, global decidability, global finite model property, and global completeness of a logic $\mathcal{L}$ are equivalent to their local versions for $\mathcal{L}^{\mathsf{E}}$. This result can be extended to many hybrid languages.

Theorem 4.1 *Let the property P be either decidability, finite model property, or axiomatic completeness, and let $\mathcal{L}$ be any sublanguage of $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$. Then $\mathcal{L}$ has P globally iff $\mathcal{L}^{\mathsf{E}}$ has P locally.*

Going back to description languages, notice that if we use $\models^{glo}$ instead of $\models^{loc}$, then basic modal logic *is* enough to encode terminological axioms, as the following equivalence holds: $\langle T, \{\}\rangle \models \phi$ iff $T^t \models^{glo} \phi^t$. By using (1) we can move to $\langle T, \{\}\rangle \models \phi$ iff $\{\mathsf{A}(T^t)\} \models^{loc} \phi^t$. And given that the local consequence relation satisfies the deduction theorem, we obtain

$$\langle T, \{\}\rangle \models \phi \text{ iff } \models \mathsf{A}(T^t) \rightarrow \phi^t.$$

By Theorem 4.1, then, we can study logical properties of *inference from non-empty knowledge bases* through *local properties of satisfiability* of languages containing E. If the logic is compact, we can perform this reduction even for infinite T-Boxes, but, most importantly, by replacing the Kleene star with the existential modality we obtain a language with a much better behaved and understood model theory.

Let us now turn to the second feature of DLs that many modal logicians find puzzling from a model-theoretic point of view: A-Box information.

The main difficulty with the extended translation $(\cdot)^t$ defined in Section 3 is in forcing propositional symbols in CPDL to behave as individuals. If we use hybrid logics instead, we can simply use *nominals*. A-Box statements can then be straightforwardly accounted for, as $a\!:\!C$ simply becomes $@_a C^t$ and $(a,b):R$ can be translated as $@_a\langle R\rangle b$. The whole idea of nominals seems to fit neatly in the DLs perspective, so much so that many ideas and techniques from hybrid logics, involving the direct use of nominals, have been already taken up by the description logic community (see, e.g., Tobies 2000).

Given the presence of the existential modality and nominals, hybrid logics are very well suited to provide a modal perspective on description languages that is able to deal with full terminological and assertional reasoning. Consider the following translation $\cdot^h$ taking concepts, terminological axioms and assertions to hybrid formulas:

$$\begin{array}{rlrl}
(C_i)^h &= p_i,\ (C_i \text{ atomic}) & (\{a_1,\ldots,a_n\})^h &= a_1 \vee \cdots \vee a_n\\
(\neg C)^h &= \neg(C^h) & (\exists R.\{a\})^h &= \langle R\rangle a\\
(C \sqcap D)^h &= C^h \wedge D^h & (C \sqsubseteq D)^h &= \mathsf{A}(C^h \rightarrow D^h)\\
(\exists R.C)^h &= \langle R\rangle C^h & (a\!:\!C)^h &= @_a C^h\\
(\exists R^{-1}.C)^h &= \langle R^{-1}\rangle C^h & ((a,b)\!:\!R)^h &= @_a\langle R\rangle b.
\end{array}$$

Theorem 4.2 *Let $\Sigma = \langle T, A\rangle$ be a knowledge base in $\mathcal{ALCOI}$, and ϕ a terminological axiom or an assertion, then $\langle T, A\rangle \models \phi$ iff $\models (\bigwedge_{\psi\in T}\psi^h \wedge \bigwedge_{\psi\in A}\psi^h) \rightarrow \phi^h$.*

The proof is obvious (and the connection between the two languages is stronger than with CPDL), as any model of $\langle T, A\rangle$ and ϕ can be viewed as a model of $(\bigwedge_{\psi\in T}\psi^h \wedge \bigwedge_{\psi\in A}\psi^h) \rightarrow \phi^h$ and vice versa.

Hybrid logics have already been proposed as natural counterparts of description languages. Blackburn and Tzakova (1998) propose using hybrid languages to embed description logics, highlighting the connection between assertional information and nominals, and the use of the existential modality to encode terminological axioms. But they introduce undecidable hybrid languages for this account, arguing that the increase in expressive power of these languages is an advantage. Instead, our translation tries to remain as faithful as possible to the original description language, and pays special attention to decidability issues.

What kind of expressive power is needed to encode the different languages and reasoning tasks? For example, the existential modality is required only for translating terminological axioms, while @ is only used for assertions. The following lists a number of relevant hybrid languages:

- $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$, in which the full translation of $\mathcal{ALCOI}$ with non-empty T-Boxes and A-Boxes can be made.
- $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$, in which only inferences in terms of empty T-Boxes can be performed.
- $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$, in which we only allow nominals to appear as subindices of @ and in the construction $@_a\langle R\rangle b$ or $@_a\langle R^{-1}\rangle b$, and hence we can translate neither the one-of operator $\mathcal{O}$ nor role fillers $\mathcal{B}$.
- $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$, the "local" version of the language above, where we work with empty T-Boxes.

We have defined each of the logics above to be expressive enough to permit the encoding of certain specific DLs. But it is also important to determine if and how we have *extended* the expressive power of the source language with the move into these hybrid languages. The general answer is: we have incorporated Boolean structure into the knowledge base, and allowed explicit interaction among T-Box definitions, A-Box assertions and concepts.

As an example, we will show how to recast satisfiability of a formula in $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$, in terms of satisfiability in the corresponding description logic. It will be useful to first discuss a normal form for hybrid formulas. Take a formula ϕ in $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$. We start by translating ϕ into propositional normal form. Then we use the following equivalences to "push out" the E and A operators from inside the other modalities (we use Q to range over E and A):

$$\begin{array}{rcl|rcl} [R]\mathsf{Q}\psi & \leftrightarrow & [R]\bot \vee \mathsf{Q}\psi & @_s\mathsf{Q}\psi & \leftrightarrow & \mathsf{Q}\psi \\ [R](\theta \vee \mathsf{Q}\psi) & \leftrightarrow & [R]\theta \vee \mathsf{Q}\psi & @_s(\theta \vee \mathsf{Q}\psi) & \leftrightarrow & @_s\theta \vee \mathsf{Q}\psi \\ [R](\theta \wedge \mathsf{A}\psi) & \leftrightarrow & [R]\theta \wedge [R]\mathsf{A}\psi & @_s(\theta \wedge \mathsf{Q}\psi) & \leftrightarrow & @_s\theta \wedge \mathsf{Q}\psi \\ [R](\theta \wedge \mathsf{E}\psi) & \leftrightarrow & [R]\bot \vee ([R]\theta \wedge \mathsf{E}\psi) & & & \end{array}$$

Similar equivalences hold for the dual modalities $\langle R\rangle$ (@ is self dual). For pushing out @ we have

$$\begin{array}{rcl|rcl} [R]@\psi & \leftrightarrow & [R]\bot \vee @_s\psi & @_s@_t\psi & \leftrightarrow & @_t\psi \\ [R](\theta \vee @_s\psi) & \leftrightarrow & [R]\theta \vee @_s\psi & @_s(\theta \vee @_t\psi) & \leftrightarrow & @_s\theta \vee @_t\psi \\ [R](\theta \wedge @_s\psi) & \leftrightarrow & [R]\theta \wedge [R]@_s\psi & @_s(\theta \wedge @_t\psi) & \leftrightarrow & @_s\theta \wedge @_t\psi. \end{array}$$

And similarly for the @ operators appearing under $\langle R\rangle$. We arrive then to the following result.

Proposition 4.3 *Let ϕ be a hybrid formula, then ϕ is equivalent to a formula ϕ' where subformulas of the form $\mathsf{E}\psi$, A and $@_i\psi$ (if any) occur* *only at modal depth 0. In particular, ϕ' can be taken to be*

$$\bigwedge_{l\in L}\left(\bigvee_{m\in M}\mathsf{A}\rho_{(l,m)}\vee\mathsf{E}\sigma_l\vee\bigvee_{i\in\mathsf{NOM}(\phi)}@_i\nu_{(l,i)}\vee\tau_l\right),$$

for some (possibly empty) index sets L, M, where $\rho_{(l,m)}$, σ_l, $\nu_{l,i}$ and τ_l contain neither E *nor* @.

By allowing negations in the T-Box we can encode validity of formulas in $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$ as instance checking as follows. Define *Boolean knowledge bases* as pairs $\Sigma = \langle T, A\rangle$ where T is a set of Boolean combinations of terminological axioms, and A a set of Boolean combinations of assertions. In the description logic community, Boolean knowledge bases have been considered in the setting of the CLASSIC system (Borgida et al. 1989), while authors with a modal logic background, such as Wolter and Zakharyaschev (2000), have also considered this kind of knowledge bases.

Take $\phi \in \mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$ in the normal form described in Proposition 4.3. For $l \in L$, define the knowledge base $\Sigma^l_\phi = \langle T^l_\phi, A^l_\phi\rangle$ by putting

$$\begin{aligned} T^l_\phi &= \{\neg(\top \sqsubseteq \rho^{h^{-1}}_{(m,l)}) \mid m \in M\} \cup \{\top \sqsubseteq \neg\sigma^{h^{-1}}_l\}, \text{ and} \\ A^l_\phi &= \{i{:}\neg\nu^{h^{-1}}_{(l,i)} \mid i \in \mathsf{NOM}(\phi)\}, \end{aligned}$$

where the mapping $\cdot^{h^{-1}}$ is the backwards translation from the hybrid language into $\mathcal{ALCOI}$ that sends Boolean and modal operators to the corresponding description logic ones and using singleton one-of sets $\{i\}$ for translating nominals.

Theorem 4.4 *For any formula ϕ in $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$, let $a \notin \mathsf{NOM}(\phi)$, then ϕ is valid iff for all $l \in L$, $\Sigma^l_\phi \models a{:}\tau^{h^{-1}}_l$.*

Interestingly, even if we allow Boolean knowledge bases, we cannot recast validity of hybrid formulas as inference in terms of a *unique* knowledge base. This is because the separation between terminological axioms, assertions and simple concepts imposes syntactic restrictions which don't exist in hybrid logic. Trivially, if the index set L above is a singleton, then a unique knowledge base is sufficient. I.e., we can precisely characterize the fragment of $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$ that captures the expressivity of $\mathcal{ALCOI}$ with Boolean knowledge bases.

As we will see in Section 5, allowing the extra flexibility that Boolean knowledge bases offer does not modify the complexity class in which the reasoning tasks fall (for the languages we are considering), but it increases expressive power and has an impact on meta-logical properties like interpolation and Beth definability.

5 Exploiting the Connection

The links between hybrid logics and DLs discussed in the previous sections are so strong, that we can immediately start harvesting by interpreting results from one field in the other. This is what we will do now, from a number of perspectives, including complexity, expressive power and meta-logical properties.

5.1 Complexity

We start by exporting complexity results for hybrid logics to DLs. We need to pay attention to the difference between local and global notions. For a modal language, we can distinguish between the local-*Sat* problem (given a formula ϕ, does there exist a model $\mathcal{M}$ and $m \in M$ with $\mathcal{M}, m \Vdash \phi$?), and the global-*Sat* problem (is there a model $\mathcal{M}$ with $\mathcal{M} \Vdash \phi$?). If the logic contains the E modality, the problems coincide, as we argued in Section 4.

First, we consider the "pure future" fragments of the hybrid languages defined in Section 4, i.e., we only consider formulas without the $\langle R^{-1} \rangle$ operator. The local-*Sat* problem for $\mathcal{H}_{\mathsf{N}}(@)$ is PSPACE-complete (Areces et al. 1999b). This result also settles the complexity of the language $\mathcal{H}(@, @\Diamond)$, because this language contains the basic modal language. As a corollary of the EXPTIME-completeness of CPDL, we obtain an EXPTIME upper bound for the local-*Sat* problem for $\mathcal{H}_{\mathsf{N}}(@, \mathsf{E})$. It follows from Spaan's (1993) results on the EXPTIME-completeness of modal logics with the existential modality, that both $\mathcal{H}(@, @\Diamond, \mathsf{E})$ and $\mathcal{H}_{\mathsf{N}}(@, \mathsf{E})$ are EXPTIME-complete.

Switching to the DL perspective, the results above imply that it is the move from empty T-Boxes to full T-Boxes that modifies complexity. And this does not depend on our extension to "Boolean" knowledge bases, as the same complexity results obtain when we restrict ourselves to standard knowledge bases. The one-of operator $\mathcal{O}$ and role fillers $\mathcal{B}$ offer more expressivity at no cost (up to a polynomial). At this very point the encoding of DLs into hybrid languages, instead of CPDL, works to our advantage, as we can identify cases with a PSPACE upper bound.

Theorem 5.1 1. *Instance checking for Boolean knowledge bases with empty T-Boxes is PSPACE-complete for the language $\mathcal{ALCO}$.*

2. *Instance checking for Boolean knowledge bases is solvable in EXPTIME (hence EXPTIME-complete) for the language $\mathcal{ALCO}$.*

Notice that we don't need to restrict to empty A-Boxes in item 1, and recall that the complexity results for instance checking extend to all standard reasoning tasks like knowledge base consistency, or subsumption checking.

What about the $\langle R^{-1}\rangle$ operator? Adding just *one* nominal to basic temporal logic moves the complexity of the local-*Sat* problem from PSPACE-hard to EXPTIME-hard. The known EXPTIME upper bound for CPDL plus nominals and E (Areces et al. 1999b) also covers the language $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$; hence, the local-*Sat* problems of $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$, $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ and $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$ are EXPTIME-complete.

A PSPACE upper bound for $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$ is easy to establish by using the fact that @ operators need only appear at modal depth 0.[1] We give a sketch of the proof. To avoid confusion we will write $@_i\langle R_r\rangle j$ as $R_r(i,j)$. Let

$$\phi = \bigwedge_{l\in L}\left(\bigvee_{i\in\mathsf{NOM}} @_i\nu_{(l,i)} \vee \bigvee T_l \vee \sigma_l\right),$$

where each T_l is a collection of formulas of the form $R_r(i,j)$ or $\neg R_r(i,j)$, and $\nu_{(l,i)}, \sigma_l$ contain neither @ nor nominals. As PSPACE = NPSPACE, non-deterministically choose from each conjunct of ϕ the disjunct satisfied by a model of ϕ. Collect these choices in a set called CHOICE. Now, for each i, let $S_i = \{\phi \mid @_i\phi \in \mathsf{CHOICE}\}$, and create a polynomial model satisfying S_i at the point m_i (notice that all formulas in S_i are basic temporal formulas and hence a PSPACE model can be constructed). Similarly, create a polynomial model for all formulas in CHOICE which are not @-formulas. Let $\mathcal{M}$ be the disjoint union of all these models. Finally, if $R_r(i,j) \in \mathsf{CHOICE}$, add the pair (m_i, m_j) to R_r. The model of ϕ obtained in this way has size polynomial in $|\phi|$.

With the translation into CPDL it would of course be impossible to evaluate the difference made by the presence or absence of the $\langle R^{-1}\rangle$ operator in terms of complexity.

Theorem 5.2

1. *Instance checking for Boolean knowledge bases with empty T-Boxes is solvable in PSPACE (hence PSPACE-complete) for the language $\mathcal{ALCBI}$.*
2. *Instance checking for knowledge bases with empty T- and A-Boxes is EXPTIME-hard for the language $\mathcal{ALCOI}$.*
3. *Instance checking for Boolean knowledge bases is solvable in EXPTIME (hence EXPTIME-complete) for the language $\mathcal{ALCOI}$.*

[1] Notice that we cannot directly use the normal form of Proposition 4.3 here, as the formula ϕ' in normal form can be exponentially larger than the original ϕ (already the first step in the transformation is to translate ϕ in propositional normal form). But pulling out the @ modality with the help of new propositional symbols is simple.

The complexity results listed so far were based on importing hybrid logic results into DL. The EXPTIME-hardness result for $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle)$ (basic temporal logic with at least one nominal) contrast sharply with the good complexity behavior of $\mathcal{H}_{\mathsf{N}}(@)$. For example, if we move to the class of transitive models, even $\mathcal{H}_{\mathsf{N}}(@, \mathsf{E})$ is PSPACE-complete (meaning that there are PSPACE algorithms even for inference from non-empty T-Boxes, when restrictions are set on roles), while $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle)$ remains obstinately in EXPTIME. Results concerning the complexity of hybrid logics in different classes of models are investigated in (Areces et al. 2000). When translated in DL terms, one of the main results in that paper implies that instance checking for Boolean knowledge bases in $\mathcal{ALCOI}$ can be solved in PSPACE if we consider only transitive trees as models.

Here we have mainly exported complexity results from hybrid logics to description logics, but of course the other direction is also open: known complexity results from DLs can be translated in hybrid terms. For example, little is known about the extension of hybrid languages with counting, while counting is widely used in DLs. Further, there is a "folklore" result which states that instance checking for $\mathcal{ALC}$ with T-Boxes restricted to simple and acyclic terminological axioms is PSPACE-complete; this implies that when syntactic restrictions are imposed on the use of E, we can avoid EXPTIME-hardness for the local-*Sat* problem of $\mathcal{H}(@, @\Diamond, \mathsf{E})$. In this respect, Lutz (1999a, 1999b) provides the first detailed complexity analysis of inference from simple, acyclic T-Boxes. Interestingly, the restriction to simple, acyclic T-Boxes does not always preserve complexity: instance checking in $\mathcal{ALCF}$ ($\mathcal{ALC}$ extended with features, feature agreement and feature disagreement) is PSPACE-complete for empty T-Boxes, but it turns NEXPTIME-complete even when only simple, acyclic T-Boxes are allowed.

5.2 Expressive Power

We now consider expressive power, and we do so by taking advantage of hybrid bisimulations. Bisimulations are binary relations on the domain of hybrid models. Kurtonina and de Rijke (1999) provide a detailed analysis of the expressive power of DLs by means of (bi-)simulations, but their results only address the expressive power of *concepts*. In this section, we will instead study the expressive power offered by full knowledge bases.

Let $\mathcal{M} = \langle M, \{R_r^{\mathcal{M}}\}, V^{\mathcal{M}}\rangle$ and $\mathcal{N} = \langle N, \{R_r^{\mathcal{N}}\}, V^{\mathcal{N}}\rangle$ be two hybrid models. For $i \in \mathsf{NOM}$, let $i^{\mathcal{M}}$ be the denotation of i in $\mathcal{M}$ and similarly for $i^{\mathcal{N}}$. Let $\sim$ be a non-empty binary relation on $M \times N$, and consider the following properties on $\sim$ in addition to the conditions for bisimulation for a basic temporal language (Blackburn et al. 2001):

(@) For all nominals i in NOM, $i^{\mathcal{M}} \sim i^{\mathcal{N}}$.

(@◇) Let i, j be nominals in NOM, then $R_r(i^{\mathcal{M}}, j^{\mathcal{M}})$ iff $R_r(i^{\mathcal{N}}, j^{\mathcal{N}})$.

(E) $\sim$ is total and surjective.

A bisimulation $\sim$ for a basic temporal language is a $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$-bisimulation if it satisfies the conditions **(@)** and **(@◇)**. Further, $\sim$ is a $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$-bisimulation if bisimilar states agree on all nominals (and in this case **(@◇)** can be derived from the others). $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$- and $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$-bisimulations are obtained from $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$- and $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$-bisimulations, respectively, by requiring, in addition, condition **(E)**.

Proposition 5.3 *Let $\mathcal{H}$ be one of $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$, $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$, $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ and $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$. Let $\mathcal{M} = \langle M, \{R_r^{\mathcal{M}}\}, V^{\mathcal{M}}\rangle$ and $\mathcal{N} = \langle N, \{R_r^{\mathcal{N}}\}, V^{\mathcal{N}}\rangle$, and let $\sim$ be an $\mathcal{H}$-bisimulation between $\mathcal{M}$ and $\mathcal{N}$. Then for $m \in M, n \in N$, and ϕ in $\mathcal{H}$, $m \sim n$ implies $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{N}, n \Vdash \phi$.*

For two logics $\mathcal{H}$ and $\mathcal{H}'$ we write $\mathcal{H} \preccurlyeq \mathcal{H}'$ if there is a translation $\cdot^* : \mathcal{H} \to \mathcal{H}'$, such that for each formula ϕ, for any model $\mathcal{M}$ and state m in $\mathcal{M}$, $\mathcal{M}, m \Vdash \phi$ if and only if $\mathcal{M}, m \Vdash \phi^*$. We write $\mathcal{H} \prec \mathcal{H}'$ if $\mathcal{H} \preccurlyeq \mathcal{H}'$ and not $\mathcal{H}' \preccurlyeq \mathcal{H}$. It is immediate that $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond) \preccurlyeq \mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$ and $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E}) \preccurlyeq \mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$. More interestingly, each of the relations is actually strict, which can be shown by means of bisimulations. In DL terms this means, for instance, that the one-of operator $\mathcal{O}$ does increase the expressive power of the language, both with full and empty T-Boxes.

The relation between $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$ and $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ is more complex. Using bisimulations, we can prove both that $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @) \not\preccurlyeq \mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ and $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E}) \not\preccurlyeq \mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$. Nevertheless, $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ is at least as expressive as $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$ if we are only interested in satisfiability (and not in the existence of an equivalent formula).

Proposition 5.4 *There exists a translation $\cdot^*$ from $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$ into $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ such that such that for all $\phi \in \mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$, ϕ is satisfiable iff ϕ^* is satisfiable.*

Such comparisons of the expressive power of hybrid logics easily translate to description languages. For two description languages $\mathcal{L}_1$ and $\mathcal{L}_2$, define $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ if for any knowledge bases Σ in $\mathcal{L}_1$ there is a knowledge base Σ' in $\mathcal{L}_2$ such that for all interpretations $\mathcal{I}$, $\mathcal{I} \models \Sigma$ iff $\mathcal{I} \models \Sigma'$. Now, the formulas used to separate the languages can easily be recast as assertions or terminological definitions, and similarly for the translation used in the proof of Proposition 5.4.

The notions of bisimulation we have defined not only separate the fragments of first-order logic which corresponds to the hybrid logics we have been discussing, they also *characterize* them. For $\mathcal{H}$ any of our hybrid languages, we say that a first-order formula $\alpha(x)$ in the first-order language over $\langle \mathsf{REL} \cup \{P_j \mid p_j \in \mathsf{PROP}\}, \mathsf{NOM}, \{x, y\}\rangle$ is *invariant for $\mathcal{H}$-bisimulations* if for all models $\mathcal{M}$ and $\mathcal{N}$, and all states m in $\mathcal{M}$, n in $\mathcal{N}$, and all $\mathcal{H}$-bisimulations $\sim$ between $\mathcal{M}$ and $\mathcal{N}$ such that $m \sim n$, we have $\mathcal{M} \models \alpha(x)[m]$ iff $\mathcal{N} \models \alpha(x)[n]$.

Theorem 5.5 *Assume that $\mathcal{H}$ is one of the following: $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$, $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @)$, $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond, \mathsf{E})$ or $\mathcal{H}_{\mathsf{N}}(\langle R^{-1}\rangle, @, \mathsf{E})$. Then, a first-order formula $\alpha(x)$ over the signature $\langle \mathsf{REL} \cup \{P_j \mid p_j \in \mathsf{PROP}\}, \mathsf{NOM}, \{x, y\}\rangle$ is invariant for $\mathcal{H}$-bisimulations iff it is equivalent to the hybrid translation of a hybrid formula in $\mathcal{H}$.*

We have only scratched the surface on expressivity issues. For example, definability results for hybrid languages, like the ones presented here and those in (de Rijke 1992, Gargov and Goranko 1993, de Rijke and Sturm 2001) shed light on the kinds of models that can be captured by means of knowledge bases of a given description language. More generally, Gargov and Goranko (1993) discuss transfer results when moving from basic modal languages to languages with nominals, while Goranko and Passy (1992) give a similar analysis for the extension with the existential modality. These results are closely related to the move from empty knowledge bases to non-empty A- and T-Boxes, respectively.

5.3 Interpolation and Beth Definability

In (Areces et al. 1999a) results concerning the interpolation and Beth definability properties for a variety of hybrid languages are provided. What is the role of these two properties in the setting of description logics? Let's first introduce some notation. For $\Sigma = \langle T, A\rangle, \Sigma' = \langle T', A'\rangle$ two knowledge bases, let $\Sigma \cup \Sigma'$ be $\langle T \cup T', A \cup A'\rangle$, and $\Sigma[C/D]$ be the knowledge base obtained from Σ by replacing each occurrence of the concept C by D. Now, suppose that for a given knowledge base Σ the following holds,

(2) $\Sigma[C/D_1] \cup \Sigma[C/D_2] \models D_1 \doteq D_2$ for some $D_1, D_2 \notin \mathsf{CON}(\Sigma)$.

Notice that this equation need not be true for all knowledge bases Σ and concepts C. Actually, (2) implies that Σ encodes enough information concerning C to provide a complete—though not necessarily explicit—definition. Now, if the (global) Beth definability property holds for the language of Σ, then there actually exists an explicit definition of C. I.e., there is a concept D not involving C such that $\Sigma \models C \doteq D$. Given that description languages take definitions very seriously (partial and complete definitions are exactly the content of T-Boxes), the Beth definability property (i.e., the capacity of the language to turn implicit definitions into explicit ones) seems highly relevant.

There doesn't seem to be a uniform direct way of proving or disproving Beth definability. The standard approach to establishing the property is via a detour through interpolation. In first-order and modal languages, the (arrow) interpolation property implies the Beth definability property and the same relation holds for the hybrid languages we have introduced.

Hence, positive interpolation results for hybrid languages translate into nice definability properties of the corresponding description language. Unfortunately, for languages where nominals appear free in formulas, and which do not provide a binding mechanism, failure of arrow interpolation seems to be the norm. In particular, (Areces et al. 1999a) provides counter-examples to the arrow interpolation property for the basic modal language extended with nominals $\mathcal{H}_{\mathsf{N}}(@)$. The extensions of this language with the $\langle R^{-1}\rangle$ operator fare no better, and adding the E operator doesn't help either. Hence, in all these cases, the standard road to to establish Beth definability is closed for us. Interestingly, the counter-examples to arrow interpolation obtained are based on counting arguments. Because the language is not expressive enough to bound the number of successors of a given state we can establish bisimulations between points with different number of successors and use this to prove failure of the interpolation property. The language extended with counting operators (even unqualified counting) would destroy the bisimilarity and hence invalidate the counter-examples, and perhaps restore interpolation and hence Beth definability.

The case is different for $\mathcal{H}(@, @\Diamond)$ and $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$. As we will now show, we can extend the constructive method for establishing arrow interpolation presented in (Kracht 1999, Section 3.8), to handle @ and $@\Diamond$. Again, we will use the normal form introduced in Proposition 4.3.

Theorem 5.6 *$\mathcal{H}(@, @\Diamond)$ and $\mathcal{H}(\langle R^{-1}\rangle, @, @\Diamond)$ have arrow interpolation.*

Given that arrow interpolation implies global Beth definability for these languages, implicit definitions in $\mathcal{H}(@, @\Diamond)$ can be turned into explicit definitions. In an attempt to transfer this property to the description logic counterpart of $\mathcal{H}(@, @\Diamond)$, we would proceed as follows. Suppose a knowledge base $\Sigma = \langle T, A\rangle$ in $\mathcal{ALC}$ satisfies the conditions in (2). Then we can translate Σ into a theory Θ of $\mathcal{H}(@, @\Diamond)$ (as we are using global consequence this time we don't need E), and obtain $\Theta[p_C/p_{D_1}] \cup \Theta[p_C/p_{D_2}] \models^{glo} p_{D_1} \leftrightarrow p_{D_2}$. Applying Beth definability for $\mathcal{H}(@, @\Diamond)$ we obtain a formula δ such that $\Theta \models^{glo} \delta \leftrightarrow p_C$. Now, δ is an explicit definition of C, but it is in the *full* language $\mathcal{H}(@, @\Diamond)$, i.e., it might contain subformulas of the form $@_i\psi$ and $@_i\Diamond j$. Because of the syntactic restrictions imposed by the division into T- and A-Box information it will not always be possible to translate δ into a concept in $\mathcal{ALC}$. To see an example, suppose δ is of the form $@_i\nu \vee \psi$. Hence we will have that $\Sigma \models (@_i\nu \to (p_C \leftrightarrow \top)) \wedge (@_i\neg\nu \to (p_C \leftrightarrow \psi))$. That is, we obtain a definition of C that is conditional on assertional information.

More generally, we first write δ in normal form to obtain

$$\Theta \models^{glo} \left(\bigwedge_{l\in L} (\bigvee_{i\in\mathsf{NOM}} @_i\nu_{(i,l)}) \vee \tau_l \right) \leftrightarrow p_C.$$

Notice that for a hybrid formula ψ and $@_i\nu \in \mathsf{SF}(\psi)$ such that $@$ does not appear in ν, ψ is equivalent to $(@_i\nu \to \psi[@_i\nu/\top]) \wedge (@_i\neg\nu \to \psi[@_i\nu/\bot])$. By iterating this rewriting on the formula $(\bigwedge_{l\in L}(\bigvee_{i\in\mathsf{NOM}} @_i\nu_{(i,l)}) \vee \tau_l) \leftrightarrow p_C$, we finally obtain a series of definitions of C in terms of concepts of $\mathcal{ALC}$, but conditional on assertional information to be inferred from Σ.

There is an interesting connection between the Beth definability property and acyclic definitions in T-Boxes. The latter restriction was aimed at avoiding the introduction of circular concepts, i.e., concepts defined in terms of themselves. This kind of concepts, it was argued, called for some kind of fixed point semantics which would be computationally expensive (Baader 1990, Nebel 1990). But if the language has the Beth definability property, any concept implicitly defined in a knowledge base also has an explicit definition *without self reference*. Hence, considering only acyclic definitions does not carry any loss of expressivity.

6 Conclusions and Further Directions

Nearly a decade after Schild started exploring the connection between modal and description logic, we have made another step forwards in finding a precise modal logical counterpart of DLs. One of the key points of DLs is their use of non-empty T-Boxes and A-Boxes; we have shown how hybrid languages provide simple mechanisms to deal with them and to understand their inter-relations. We have illustrated some of the possibilities by means of examples. Our analysis of DLs in terms of hybrid logics has shown that we can reason with non-empty A-Boxes (and empty T-Boxes) in PSPACE; we have also shown how to capture the expressive power offered by T-Boxes, A-Boxes, and how to transfer meta-logical properties such as interpolation and Beth definability.

It is striking how description and hybrid logics are similar and different at the same time, like twin brothers raised separately. Because the connection is so tight, modal logicians can use hybrid logic both as an entrypoint to DLs, and as a means for understanding them. In particular, we can investigate meta-logical properties on the "hybrid logic side" (which appears to be more amenable to standard model-theoretic techniques), and these results can help us understand the behavior of description languages.

We have only investigated some of the possibilities of this two-way interchange, but there are many others of course. One direction, for example, concerns the classical hybrid topic of *binders and variables* (Blackburn and Seligman, 1995). We have focused on "weak" hybrid languages which remain close to the basic DLs. But a natural step from the hybrid point of view is to regard nominals not as names but as *variables* over individual states, and to add quantifiers. Undecidability quickly shows up in this setting, but syntactic restriction can restore decidability, while providing interesting new concepts when introduced in a description language. A different direction concerns *counting*. Graded or counting modalities $\langle n \rangle \phi$ restrict the number of possible successors satisfying ϕ that a state has in a model. While their theory is not so well developed, qualifying number restrictions are actively used in description formalisms, as they provide important modeling power. Recent work by Tobies (2000) provides promising new complexity results.

Different enough to make comparisons interesting, but similar enough to allow for extensive traffic of results, extensions and variations, description logics and hybrid logics form an interesting pair. We hope that this paper paves the way for further cross-fertilization.

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