Characterization Results for d-Horn Formulas,

or On formulas that are true on Dual Reduced Products

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Abstract: We provide two different model theoretic characterizations of a fragment of first-order logic which we call d-Horn formulas. This fragment is dual to the well known Horn fragment and has the same complexity for proving unsatisfiability. The method used in the characterization (syntactic translation functions between formulas which are mimicked by translation functions between models) might be applied to characterize other first-order restrictions. This paper is related to the work of Henschen and Wos (1974), but we study semantic translation together with syntactic renaming functions. We comment shortly on a number of applications of d-Horn formulas, one of which is the characterization of Context Free Grammars through a Horn $\cup$ d-Horn first-order theory.

Keywords: Horn formulas, d-Horn formulas, renaming functions, model theoretic characterizations.

0.1 Introduction

The Horn restriction of first-order logic (FO) is a relevant fragment, specially for Computer Science. For instance, the PROLOG programming language is based on the Horn fragment. It is well know that proving unsatisfiability of Horn sets is polynomial via the SLD resolution method. Horn sentences have also a neat model theoretic characterization due to Alfred Horn (1951): they are the formulas which are preserved under reduced products.

In this work we identify another restriction of FO, the d-Horn set, that possesses the same low complexity for showing unsatisfiability and a nice model theoretic characterization. The set of d-Horn formulas is precisely the set of formulas whose validity is preserved under *dual reduced products of models*, an operation we defined as a variant of the ordinary reduced product.

While not disjoint, Horn and d-Horn seem to be dual FO restrictions. Some examples of sentences in these classes are: reflexivity $(\forall x)P(x,x)$, irreflexivity $(\forall x)\neg P(x,x)$ and symmetry $(\forall xy)(P(x,y) \rightarrow P(y,x))$ are all Horn as well as d-Horn sentences; transitivity $(\forall xyz)(P(x,y) \wedge P(y,z) \rightarrow P(x,z))$ is a Horn sentence not equivalent to any d-Horn sentence; but there are also d-Horn formulas which are not Horn formulas, witness the sentence for connectedness $(\forall xy)(P(x,y) \vee P(y,x))$.

Henschen and Wos (1974) report that when a set S of FO formulas is not itself Horn (nor equivalent to any Horn set), there often exists a polynomial*renaming* of S that yields a Horn set S'. A renaming function is a syntactic translation which has the crucial property of preserving satisfiability. Since renaming functions have polynomial complexity, it is immediate to realize that proving unsatisfiability for renameable Horn sets has also polynomial complexity. In this work we concentrate on a special renaming function that maps Horn formulas to d-Horn formulas and conversely. Based on this translation function and the original model theoretic characterization result for Horn formulas we develop a model theoretic characterization of the d-Horn set.

The sort of duality explored in this work suggests studying a similar behavior for other sets of FO formulas, as for example the set of positive formulas characterized by preservation under homomorphisms.

0.2 The Propositional Case

We start with a simple result in Propositional Logic (PL) which motivates our further research in FO. Conditional sentences in PL are defined, for example, in (Chang and Keisler 1990) and they have the following model theoretic characterization:

Definition 1 (Conditional Sentences) A *conditional sentence* is a conjunction $\varphi_1 \wedge \ldots \wedge \varphi_n$ in PL such that each φ_i is either

- a propositional symbol S,
- a disjunction of negated propositional symbols $\neg S_1 \vee \ldots \vee \neg S_n$, or
- a disjunction of negated propositional symbols and a propositional symbol $\neg S_1 \vee \ldots \vee \neg S_n \vee S_{n+1}$.

Theorem 1 (Characterization of Conditional Sentences) A theory Γ of PL is preserved under intersections if and only if Γ has a set of conditional axioms.

Where "to be preserved under intersections" means that for a non empty index set I, $\bigwedge_{i \in I} \mathcal{A}_i \models \Gamma \Rightarrow \bigcap_{i \in I} \mathcal{A}_i \models \Gamma$. Since models in PL can be seen as sets of propositional variables (those which are true in the model), the concept of intersecting models is well defined.

Consider now the dual pattern: conjunction of sentences where only at most one negative propositional symbol appears in disjunction with positive propositional symbols (which we call quasi-positive sentences.) A similar characterization for this set is not hard to find.

Theorem 2 (Characterization of Quasi-Positive Sentences) A theory Γ of PL is preserved under unions if and only if Γ has a set of quasi-positive axioms.

Proof. To prove the right to left implication it suffices to show that quasi-positive sentences are preserved under unions. Namely, let φ be quasi-positive and take two models such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \varphi$ we have to show that $\mathcal{A} \cup \mathcal{B} \models \varphi$.

If φ is a positive sentence then clearly from $\mathcal{A} \models \varphi$ we can infer $\mathcal{A} \cup \mathcal{B} \models \varphi$. Suppose then that $\varphi = (\neg S_i \vee \psi)$ for S_i a propositional symbol and ψ a positive sentence. There are two cases:

- If $S_i \in \mathcal{A}$ then $\mathcal{A} \models (\neg S_i \vee \psi)$ implies $\mathcal{A} \models \psi$. Then $\mathcal{A} \cup \mathcal{B} \models \psi$, since ψ is positive, and $\mathcal{A} \cup \mathcal{B} \models \neg S_i \vee \psi$.
- If $S_i \notin \mathcal{A}$ then again there are two cases
  - If $S_i \in \mathcal{B}$ then $\mathcal{B} \models \psi$. Then $\mathcal{A} \cup \mathcal{B} \models \psi$, and $\mathcal{A} \cup \mathcal{B} \models \neg S_i \vee \psi$.
  - If $S_i \notin \mathcal{B}$ then $S_i \notin \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B} \models \neg S_i$. Hence $\mathcal{A} \cup \mathcal{B} \models \neg S_i \vee \psi$.

Let's see now the hard direction. Suppose Γ is preserved under unions. Let Δ the set of quasi-positive consequences of Γ. It suffices to show that if $\mathcal{B} \models \Delta$ then $\mathcal{B} \models \Gamma$.

Let $\mathcal{B}$ be a model for Δ. For each $S_i \in \mathcal{B}$ we define Σ_{S_i} the set of sentences $S_i \wedge \psi$ where ψ is negative, that are true in $\mathcal{B}$. Notice that the finite conjunction of elements of Σ_{S_i} is equivalent to a sentence in Σ_{S_i}.

Let $\varphi \in \Sigma_{S_i}$. Clearly $\neg\varphi$ is equivalent to a quasi-positive sentence θ that is falsified in $\mathcal{B}$. Then, $\neg\varphi$ is not a consequence of Γ, therefore $\Gamma \cup \{\varphi\}$ is satisfiable. Hence, $\Gamma \cup \Sigma_{S_i}$ is satisfiable.

Let $\mathcal{A}_{S_i}$ be a model of $\Gamma \cup \Sigma_{S_i}$. If $S_i \in \mathcal{B}$ then $S_i \in \mathcal{A}_{S_i}$ and $\mathcal{B} \subseteq \bigcup_{S_i \in \mathcal{B}} \mathcal{A}_{S_i}$.

We show $\bigcup_{S_i \in \mathcal{B}} \mathcal{A}_{S_i} \subseteq \mathcal{B}$. Suppose $S_j \notin \mathcal{B}$, then $\mathcal{B} \models \neg S_j$ and $S_i \wedge \neg S_j \in \Sigma_{S_i}, \forall S_i \in \mathcal{B}$. Thus, $S_j \notin \mathcal{A}_{S_i}, \forall S_i \in \mathcal{B}$ and $S_j \notin \bigcup_{S_i \in \mathcal{B}} \mathcal{A}_{S_i}$. Hence $\mathcal{B} = \bigcup_{S_i \in \mathcal{B}} \mathcal{A}_{S_i}$.

Now, each $\mathcal{A}_{S_i}$ is a model of $\Gamma \cup \Sigma_{S_i}$ and, a fortiori, a model of Γ. However, Γ is preserved by unions, then $\bigcup_{S_i \in \mathcal{B}} \mathcal{A}_{S_i}$ is a model of Γ. As $\mathcal{B} = \bigcup_{S_i \in \mathcal{B}} \mathcal{A}_{S_i}$, $\mathcal{B}$ is a model of Γ. □

This result is no more than a simple exercise in basic model theory. But what is interesting is that given the characterization for conditional sentences, we can give a much simpler proof, by using *translation functions*.

Definition 2 (Translation Function $t^{\mathcal{L}}$) Let $t^{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{L}$ be defined recursively as:

$t^{\mathcal{L}}(S) = \neg S$, for S a propositional symbol.
$t^{\mathcal{L}}(\neg S) = S$, for S a propositional symbol.
$t^{\mathcal{L}}(\varphi \vee \psi) = t^{\mathcal{L}}(\varphi) \vee t^{\mathcal{L}}(\psi)$.
$t^{\mathcal{L}}(\varphi \wedge \psi) = t^{\mathcal{L}}(\varphi) \wedge t^{\mathcal{L}}(\psi)$.

It is clear that if φ is a conditional sentence then $t^{\mathcal{L}}(\varphi)$ is a quasi-positive sentence, and vice versa. Suppose furthermore that we define the following translation between PL models.

Definition 3 (Translation Function $t^{\mathcal{M}}$) Let $t^{\mathcal{M}} : \mathcal{M} \mapsto \mathcal{M}$ be defined simply as $t^{\mathcal{M}}(\mathcal{A}) = I\!\!P \backslash \mathcal{A}$, for $I\!\!P$ the set of propositional symbols.

Again, by the fact that PL models are sets of propositional symbols the translation is well defined. We will soon drop the superscripts $^{\mathcal{L}}$ and $^{\mathcal{M}}$ when no confusion arises. An important characteristic of these translations is that they preserve validity.

Proposition 3 Let φ be a conditional sentence or a quasi-positive sentence and $\mathcal{A}$ a PL model, then $\mathcal{A} \models \varphi$ iff $t(\mathcal{A}) \models t(\varphi)$.

Now we can reprove the characterization theorem as follows.

Theorem 4 (Characterization of Quasi-Positive Sentences A theory Γ of PL is preserved under unions if and only if Γ has a set of quasi-positive axioms.

Proof.

$\Rightarrow$) Let Δ be the set of quasi-positive axioms for Γ. Let also $\mathcal{A}_i, i \in I$ be PL models such that $\mathcal{A}_i \models \Gamma$. By definition, $t(\Delta)$ is a set of conditional sentences. By Proposition 3 above, $t(\mathcal{A}_i) \models t(\Delta)$. By Theorem 1, $\bigcap_{i\in I} t(\mathcal{A}_i) \models t(\Delta)$. Iff, $t(\bigcap_{i\in I} t(\mathcal{A}_i)) \models t(t(\Delta))$. Iff $\bigcup_{i\in I} \mathcal{A}_i \models \Delta$.

$\Leftarrow$) Now, let Γ be preserved under unions. We claim that $t(\Gamma)$ is preserved under intersections. To prove it, take $I, \mathcal{A}_i, i \in I$ such that $\mathcal{A}_i \models t(\Gamma)$. Iff $t(\mathcal{A}_i) \models \Gamma$. As Γ is preserved under unions, $\bigcup_{i\in I} t(\mathcal{A}_i) \models \Gamma$, iff $t(\bigcup_{i\in I} t(\mathcal{A}_i)) \models t(\Gamma)$. That is, $\bigcap_{i\in I} \mathcal{A}_i \models t(\Gamma)$. Applying now Theorem 1 we have that $t(\Gamma)$ has a set of conditional axioms Δ. But then $t(\Delta)$ is a set of quasi-positive axioms for Γ. □

If attention is given to the definition of conditional sentences, it will be noted that they are "the Horn formulas of PL" while we will define d-Horn formulas as the FO equivalent of quasi-positive sentences. Furthermore, the techniques used in the last characterization result for quasi-positive sentences can be lifted (mutatis mutandis) directly to FO. This is the topic of the next section.

0.3 Horn and d-Horn Formulas

In this section $\mathcal{L}$ denotes a first-order language with the usual notational conventions. We use $\equiv$ as the identity symbol. A first-order *model* is a tuple $\mathcal{A} = \langle A, \{R_i \mid i \in I_1\}, \{f_i \mid i \in I_2\}, \{c_i \mid i \in I_3\}\rangle$ such that A is a non-empty domain, $R_i, i \in I_1$ are relations over A, $f_i, i \in I_2$ functions on A and $c_i, i \in I_3$ constants in A. We start by stating the characterization of Horn formulas as presented in (Chang and Keisler 1990).

Definition 4 (Horn Formulas) A formula φ of $\mathcal{L}$ is said to be a *basic Horn formula* iff φ is a disjunction of formulas θ_i, $\varphi = (\theta_1 \vee \ldots \vee \theta_m)$ where at most one of the formulas θ_i is an atomic formula, the rest being negations of atomic formulas.

Horn formulas are built up from basic Horn formulas with the connectives $\wedge$, $\exists$ and $\forall$. A *Horn sentence* is a Horn formula with no free variables.

The characterization uses the notion of reduced products.

Definition 5 (Filter, Ultrafilter and Reduced Product) Let I be a nonempty set, and $\mathcal{P}(I)$ the set of all subsets of I. A *filter* D over I is a set $D \subseteq \mathcal{P}(I)$ such that:

i) $I \in D$.
ii) If $X, Y \in D$ then $X \cap Y \in D$.
iii) If $X \in D$ and $X \subseteq Z \subseteq I$ then $Z \in D$.

D is a *proper filter* iff it is not the improper filter $\mathcal{P}(I)$. D is said to be an *ultrafilter* over I iff D is a filter over I such that for all $X \in \mathcal{P}(I)$, $X \in D$ iff $X^c \notin D$ (ultrafilters are always proper.) Given models $\mathcal{A}_i, i \in I$, the *reduced product over a proper filter* D (not. $\Pi_D \mathcal{A}_i$) is the model described as follows:

i) The domain is $\Pi_D A_i$ the reduced product of A_i modulo D.
ii) Let R be an n-ary relation symbol. The interpretation of R is the relation $R^{\Pi_D \mathcal{A}_i}(f^1_D \ldots f^n_D)$ iff $\{i \in I \mid R^{\mathcal{A}_i}(f^1(i) \ldots f^n(i))\} \in D$.
iii) Let F be an n-ary function symbol. Then F is interpreted by the function $F^{\Pi_D \mathcal{A}_i}(f^1_D \ldots f^n_D) = \langle F^{\mathcal{A}_i}(f^1(i) \ldots f^n(i)) : i \in I\rangle_D$.
iv) Let c be a constant. Then c is interpreted by the element $c^{\Pi_D \mathcal{A}_i} = \langle c^{\mathcal{A}_i} : i \in I\rangle_D$.

The next characterization is Theorem 6.2.5 in (Chang and Keisler 1990).

Theorem 5 (Horn Characterization) Let $\mathcal{A}_i$ for $i \in I$ be models. Let D be a proper filter on I and $f^1, \ldots, f^n$ be elements of $\Pi_{i\in I} A_i$. A formula $\varphi(x_1 \ldots x_n)$ is equivalent to a Horn formula iff

$$\{i \in I \mid \mathcal{A}_i \models \varphi[f^1(i) \ldots f^n(i)]\} \in D \Rightarrow \Pi_D \mathcal{A}_i \models \varphi[f^1_D \ldots f^n_D].$$

As in the propositional case, we can capture the set of d-Horn formulas with a proper definition of translation functions $t^{\mathcal{L}}$ and $t^{\mathcal{M}}$. We start by defining this set. Intuitively, d-Horn formulas are built exactly the same as Horn formulas but with negated atoms where Horn formulas allow for positive atoms and vice versa, but special care has to be taken with equality (which has a fixed meaning on models and cannot be "adjusted" by the model translation $t^{\mathcal{M}}$.)

Definition 6 (d-Horn Formulas) A formula φ of $\mathcal{L}$ is said to be a *basic d-Horn formula* iff $\varphi = (\theta_1 \vee \ldots \vee \theta_m)$, where at most one atomic identity formula (those of the form $t_1 \equiv t_2$) appears non negated and all atomic non identity formulas appear non negated; or (exclusive) all atomic identity formulas appear negated and at most one atomic non identity formula appears negated.

d-Horn formulas are built up from basic d-Horn formulas with the connectives $\wedge$, $\exists$ and $\forall$. *A d-Horn sentence* is a d-Horn formula with no free variables.

It is easy to see now that the sets of Horn formulas and d-Horn formulas are not disjoint. Any atomic formula is both Horn and d-Horn; so is the negation of an atomic formula, and so is an implication of a positive atomic antecedent and a positive atomic consequent. While Horn formulas include all the negative formulas (namely, formulas just involving negated atoms) d-Horn formulas include all positive formulas free of identity.

As it was said before, the formulas for reflexivity $(\forall x)P(x,x)$, irreflexivity $(\forall x)\neg P(x,x)$ and symmetry $(\forall xy)(P(x,y) \rightarrow P(y,x))$ are all Horn as well as d-Horn sentences. In contrast, the formula for connectedness $(\forall xy)(P(x,y) \vee P(y,x))$ is d-Horn but not Horn. The formula for transitivity $(\forall xyz)(P(x,y) \wedge P(y,z) \rightarrow P(x,z))$ is Horn but not d-Horn. The formula $(\forall xyz)(P(x,y) \wedge P(y,z) \rightarrow (P(x,z) \vee P(z,x))$ is neither Horn nor d-Horn. It is quite straightforward to define a translation function that takes Horn formulas to d-Horn and conversely.

Definition 7 (Translation Function $t^{\mathcal{L}}$) Let $t^{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{L}$ be defined recursively as:

$t^{\mathcal{L}}(\varphi) = \varphi$ if φ is an atomic or negated atomic identity formula.
$t^{\mathcal{L}}(\varphi) = \neg\varphi$ if φ is an atomic non identity formula.
$t^{\mathcal{L}}(\neg\varphi) = \varphi$ if φ is an atomic non identity formula.
$t^{\mathcal{L}}(\varphi \vee \psi) = t^{\mathcal{L}}(\varphi) \vee t^{\mathcal{L}}(\psi)$.
$t^{\mathcal{L}}(\varphi \wedge \psi) = t^{\mathcal{L}}(\varphi) \wedge t^{\mathcal{L}}(\psi)$.
$t^{\mathcal{L}}((\forall x)\varphi) = (\forall x)t^{\mathcal{L}}(\varphi)$.
$t^{\mathcal{L}}((\exists x)\varphi) = (\exists x)t^{\mathcal{L}}(\varphi)$.

$t^{\mathcal{L}}$ is defined in such a way that the image of a Horn formula is a d-Horn formula and vice versa. The translation function for models $t^{\mathcal{M}}$ is simpler.

Definition 8 (Translation Function $t^{\mathcal{M}}$) Let $t^{\mathcal{M}} : \mathcal{M} \mapsto \mathcal{M}$ be defined simply as $t^{\mathcal{M}}(\langle A, \{R_i \mid i \in I_1\}, \{f_i \mid i \in I_2\}, \{c_i \mid i \in I_3\}\rangle) = \langle A, \{R_i^c \mid i \in I_1\}, \{f_i \mid i \in I_2\}, \{c_i \mid i \in I_3\}\rangle$. Hence, $t^{\mathcal{M}}(\mathcal{A})$ is identical with $\mathcal{A}$ but has as relations the complements of the relations in $\mathcal{A}$.

Now we must check that satisfiability is preserved by the translations.

Proposition 6 (Satisfiability Preservation) Let $\mathcal{A}$ be a model and φ be a Horn or d-Horn formula, then $\mathcal{A} \models \varphi[a_1 \ldots a_n]$ iff $t(\mathcal{A}) \models t(\varphi)[a_1 \ldots a_n]$.

Proof. The proof is by induction on φ. The only interesting cases being when φ is atomic or negated atomic as in all the other cases t commutes over the formula.

Suppose φ is an atomic identity formula, $\varphi = (t_1 \equiv t_2)$. By the translation function $t(\varphi) = \varphi = (t_1 \equiv t_2)$. Then as $\mathcal{A}$ and $t(\mathcal{A})$ give the same interpretation to functional and constant symbols, $\mathcal{A} \models (t_1 \equiv t_2)[a_1 \ldots a_n]$ iff $t(\mathcal{A}) \models t(t_1 \equiv t_2)[a_1 \ldots a_n]$. And the same is true for a negated atomic identity formula.

Suppose $\varphi = R_i(t_1 \ldots t_m)$ for some relational symbol R_i and terms t_j. Then $t(\varphi) = \neg R_i(t_1 \ldots t_m)$. Now, $\mathcal{A} \models \varphi[a_1 \ldots a_n]$ iff $(t_1^{\mathcal{A}}[a_1 \ldots a_n] \ldots t_m^{\mathcal{A}}[a_1 \ldots a_n]) \in R^{\mathcal{A}}$ iff $(t_1^{\mathcal{A}}[a_1 \ldots a_n] \ldots t_m^{\mathcal{A}}[a_1 \ldots a_n]) \notin (R^c)^{\mathcal{A}}$. By the fact that $\mathcal{A}$ and $t(\mathcal{A})$ give the same interpretation to functional and constant symbols and the definition of $t^{\mathcal{L}}$ and $t^{\mathcal{M}}$ we obtain $t(\mathcal{A}) \models t(\varphi)[a_1 \ldots a_n]$. □

Now we can derive some useful properties of the translations.

Proposition 7

1. $t^{\mathcal{L}}$ and $t^{\mathcal{M}}$ are involutive: $t^{\mathcal{L}}(t^{\mathcal{L}}(\varphi)) = \varphi$ and $t^{\mathcal{M}}(t^{\mathcal{M}}(\mathcal{A})) = \mathcal{A}$.
2. $\models \varphi \to \psi$ iff $\models t^{\mathcal{L}}(\varphi) \to t^{\mathcal{L}}(\psi)$.
3. For any $\varphi \in \mathcal{L}$, φ is equivalent to a Horn formula iff $t^{\mathcal{L}}(\varphi)$ is equivalent to a d-Horn formula.

(Above we apply t to the formula φ where negation appears only for atoms. Every first-order formula has an equivalent which satisfies this condition.)

Proof.

1. Trivial.
2. We prove the left to right implication. The other is similar.

Suppose not. Then $\models \varphi \to \psi$ but $\not\models t^{\mathcal{L}}(\varphi) \to t^{\mathcal{L}}(\psi)$. Hence there exists some model $\mathcal{A}$ such that $\mathcal{A} \not\models t^{\mathcal{L}}(\varphi) \to t^{\mathcal{L}}(\psi)$ iff $t^{\mathcal{L}}(\mathcal{A}) \not\models t^{\mathcal{L}}(t^{\mathcal{L}}(\varphi) \to t^{\mathcal{L}}(\psi))$ iff $t^{\mathcal{L}}(\mathcal{A}) \not\models t^{\mathcal{L}}(t^{\mathcal{L}}(\varphi)) \to t^{\mathcal{L}}(t^{\mathcal{L}}(\psi))$ iff $t^{\mathcal{L}}(\mathcal{A}) \not\models \varphi \to \psi$, contradicting the hypothesis.

3. Let φ be equivalent to a Horn formula ψ. Then $\models \varphi \leftrightarrow \psi$ iff $\models t^{\mathcal{L}}(\varphi) \leftrightarrow t^{\mathcal{L}}(\psi)$ and $t^{\mathcal{L}}(\psi)$ is a d-Horn formula. □

These results let us extend the characterization results of Horn formulas to d-Horn formulas in the following way.

Theorem 8 (Indirect d-Horn Characterization) Let $\mathcal{A}_i$ for $i \in I$ be models of $\mathcal{L}$. Let D be a proper filter on I and $f^1, \ldots, f^n$ be elements of $\Pi_{i \in I} A_i$. A formula $\varphi(x_1 \ldots x_n)$ is equivalent to a d-Horn formula iff

$$\{i \in I \mid \mathcal{A}_i \models \varphi[f^1(i) \ldots f^n(i)]\} \in D \Rightarrow \Pi_D t(\mathcal{A}_i) \models t(\varphi)[f^1_D \ldots f^n_D].$$

Proof.

$\Rightarrow$) Suppose that φ is a d-Horn formula and that $\{i \in I \mid \mathcal{A}_i \models \varphi[f^1(i) \ldots f^n(i)]\} \in D$ holds. If and only if by Proposition 6 $\{i \in I \mid t(\mathcal{A}_i) \models t(\varphi)[f^1(i) \ldots f^n(i)]\} \in D$.

As $t(\varphi)$ is a Horn formula, it is preserved by reduced products: $\Pi_D t(\mathcal{A}_i) \models t(\varphi)[f^1_D \ldots f^n_D]$

$\Leftarrow$) Suppose that φ is such that whenever $\{i \in I \mid \mathcal{A}_i \models \varphi[f^1(i) \ldots f^n(i)]\} \in D$ then $\Pi_D t(\mathcal{A}_i) \models t(\varphi)[f^1_D \ldots f^n_D]$, and $\{i \in I \mid t(\mathcal{A}_i) \models t(\varphi)[f^1(i) \ldots f^n(i)]\} \in D$. Finally, $\Pi_D t(\mathcal{A}_i) \models t(\varphi)[f^1_D \ldots f^n_D]$.

Hence, $t(\varphi)$ is equivalent to a Horn formula. Now $t(t(\varphi))$ is equivalent to a d-Horn formula by Proposition 7 and by involution, φ is equivalent to a d-Horn formula. □

Using involution of the translation functions, it is possible to simplify the expression of Theorem 8 to one containing only the model translation: $t(\Pi_D t(\mathcal{A}_i)) \models t(t(\varphi))[f^1_D, \ldots, f^n_D]$ iff $t(\Pi_D t(\mathcal{A}_i)) \models \varphi[f^1_D, \ldots, f^n_D]$. The theorem would then read:

Theorem 9 (Indirect d-Horn Characterization) Let $\mathcal{A}_i$ for $i \in I$ be models of $\mathcal{L}$. Let D be a proper filter on I and $f^1, \ldots, f^n$ be elements of $\Pi_{i\in I} A_i$. A formula $\varphi(x_1 \ldots x_n)$ is equivalent to a d-Horn formula iff

$$\{i \in I \mid \mathcal{A}_i \models \varphi[f^1(i) \ldots f^n(i)]\} \in D \Rightarrow t(\Pi_D t(\mathcal{A}_i)) \models \varphi[f^1_D \ldots f^n_D].$$

Even though this last formulation involves only translations between models, we are interested now in a direct characterization of d-Horn formulas; namely, one that also eliminates the translation for models. To make a parallel with the propositional case we are now at the point where we have the characterization for $(\bigcap_{i\in I} \mathcal{A}^c_i)^c$ and we are looking for a more natural model construction like $\bigcup_{i\in I} \mathcal{A}$. To this aim we introduce dual reduced products notated as Π^*_D.

Definition 9 (Dual Reduced Product) Given models $\mathcal{A}_i, i \in I$, and D a proper filter, the *dual reduced product* $\Pi^*_D \mathcal{A}_i$ is the model described as follows:

i) The domain of $\Pi^*_D \mathcal{A}_i$ is $\Pi_D A_i$.

ii) Let R be an n-ary relation symbol. The interpretation of R in $\Pi^*_D \mathcal{A}_i$ is the relation $R^{\Pi^*_D \mathcal{A}_i}(f^1_D \ldots f^n_D)$ iff ($\exists U$ an ultrafilter) ($D \subseteq U \wedge \{i \in I \mid R_i(f^1(i) \ldots f^n(i))\} \in U$).

iii) Let F be an n-ary function symbol. Then F is interpreted in $\Pi^*_D \mathcal{A}_i$ by the function $F^{\Pi^*_D \mathcal{A}_i}(f^1_D \ldots f^1_D) = \langle F_i(f^1(i) \ldots f^n(i)) : i \in I\rangle_D$.

iv) Let c be a constant. Then c is interpreted by the element $c^{\Pi^*_D \mathcal{A}_i} = \langle a_i : i \in I\rangle_D$.

Dual reduced products are exactly like reduced products in their clauses for universes, functions and constants. What changes is the condition for the relations (this is in accordance to the translation between models we were using.) Perhaps surprisingly, the definition of dual reduced products is existential in nature ("there exists an ultrafilter U ..."), but if we consider again the propositional case we see that for an element to be in the union set, it needs to be just in one of the sets, contrasting with the universal definition of intersection.

To prove that this construction is the one we where looking for, we have to check that it coincides with the construction obtained through the translation function.

Proposition 10 Let $\mathcal{A}_i$ be models for $\mathcal{L}$, D be a proper filter on I, then $\Pi^*_D\mathcal{A}_i = t(\Pi_D t(\mathcal{A}_i))$.

Proof. As the translation does not change the universe of the model, or the interpretation of function and constant symbols these elements are in $t(\Pi_D t(\mathcal{A}_i))$ the same as in $\Pi_D\mathcal{A}_i$ and the same is true for $\Pi^*_D\mathcal{A}_i$.

It rests to check the interpretation of the relation symbols.

Suppose that R is an n-ary relational symbol of $\mathcal{L}$. We have to check that given $f^1, \ldots, f^n \in \Pi^*_D A_i$, $R(f^1_D \ldots f^n_D) \in \Pi^*_D\mathcal{A}_i$ iff $(Rf^1_D \ldots f^n_D) \in t(\Pi_D t(\mathcal{A}_i))$.

Suppose that $R(f^1_D \ldots f^n_D) \in t(\Pi_D t(\mathcal{A}_i))$ iff $R(f^1_D \ldots f^n_D) \notin \Pi_D t(\mathcal{A}_i)$ iff $\{i \in I \mid R^{t(\mathcal{A}_i)}(f^1(i) \ldots f^n(i))\} \notin D$ iff $\{i \in I \mid (R_i)^c(f^1(i) \ldots f^n(i))\} \notin D$ iff $\{i \in I \mid \neg R_i(f^1(i) \ldots f^n(i))\} \notin D$.

$\Leftarrow$) But then it is consistent to extend D to a set D^{Ext} in such a way as to get $\{i \in I \mid R_i(f^1(i) \ldots f^n(i))\} \in D^{Ext}$ and furthermore, we can make this extension maximal and take an ultrafilter U.

$\Rightarrow$) Now suppose there exists an ultrafilter U extending D such that $\{i \in I \mid R_i(f^1(i) \ldots f^n(i))\} \in U$, we want to prove that $\{i \in I \mid \neg R_i(f^1(i) \ldots f^n(i))\} \notin D$.

Suppose not, then as U extends D we have $\{i \in I \mid \neg R_i(f^1(i) \ldots f^n(i))\} \in U$. But by hypothesis $\{i \in I \mid R_i(f^1(i) \ldots f^n(i))\} \in U$ and these two sets are complementary, arriving to a contradiction with the choice of U as an ultrafilter. □

It would be very rewarding to find a direct proof for d-Horn characterization in terms of the dual reduced product operation, hopefully simpler than the proofs known for Horn characterization. Such a proof could provide an indirect Horn characterization following the same process used in this work but from d-Horn to Horn. The "hope" for a simpler proof comes from the similarities of the d-Horn fragment (without equality) with the positive formulas (remember that the propositional set was called quasi-positive) and the easy characterization result of positive formulas via homomorphisms.

The dual reduced product of a set of models is a new model theoretic construction that might have interesting properties besides those presented in this work. We will briefly comment on the connections between reduced products and dual reduced products. It is easy to observe that the satisfaction of a formula $R(t_1 \dots t_m)$ in a given reduced product $\Pi_D \mathcal{A}_i$ and valuation $[a_1 \dots a_n]$ implies satisfaction in the corresponding dual reduced product.

Proposition 11 Let $\mathcal{A}_i, i \in I$ be models, D be a proper filter on I, f_1, $\dots, f_m$ be elements of $\Pi_D A_i$ then $R^{\Pi_D \mathcal{A}_i}(t_1 \dots t_n)[f_D^1 \dots f_D^m]$ implies $R^{\Pi^*_D \mathcal{A}_i}(t_1 \dots t_n)[f_D^1 \dots f_D^n]$.

The proof only relies on the fact that every proper filter can always be extended to an ultrafilter. It is interesting to check when the converse holds. Apparently, only when D is an ultraproduct, which is a trivial case. Suppose we define a relation $E(I, D)$, where I is a set and D is a proper filter on I and that reflects the property we are studying now, namely, that for this specific proper filter D on set I, the constructions Π_D and Π^*_D are equivalent. This condition is really a condition about the relations because the only difference between the two constructions is how the relational symbols in the language are interpreted. Actually, $E(D, I)$ iff $(\forall \{A_i\}_{i \in I})$ $(\forall n \in I\!N)$ $(\forall R)$ $(\forall f_D^1, \dots f_D^n \in \Pi_D A_i) R^{\Pi_D \mathcal{A}_i}(f_D^1 \dots f_D^n) \leftrightarrow R^{\Pi^*_D \mathcal{A}_i}(f_D^1 \dots f_D^n)$.

We know that $R^{\Pi_D A_i}$ is always included in $R^{\Pi^*_D A_i}$. So in order to satisfy $E(D, I)$ it is necessary and sufficient that $R^{\Pi^*_D A_i} \subseteq R^{\Pi_D A_i}$ holds. But as we saw in the proof of Proposition 10 for this to be true we need D to be an ultrafilter. Perhaps, setting further conditions on R or $\{\mathcal{A}_i\}, i \in I$ we can obtain a less strict condition on D. These issues are outside the scope of this work.

0.4 The Interest of the d-Horn Restriction

In this section we comment on the possible significance of the d-Horn restriction.

As we already remarked, while the expressive power of the d-Horn set is different from Horn's, they both enjoy the same low complexity for unsatisfiability. Determining unsatisfiability of a Horn set is polynomial (Chang and Lee 1973, Schöning 1989). As Henschen and Wos proved in their original paper (Henschen and Wos 1974) Horn renamable sets inherit the same virtue, and our synctactic translation mapping Horn to d-Horn formulas (and viceversa) satisfies the conditions of a renaming function. The complexity upper bound can be also established directly, given the linear complexity of the translation function and Corollary 6. Let Γ be a d-Horn theory and φ a d-Horn formula. $\Gamma \models \varphi$ holds iff

$t(\Gamma) \models t(\varphi)$ holds. Given that $t(\Gamma)$ is equivalent to a Horn theory and $t(\varphi)$ a Horn formula, a refutaion can be found in polynomial time by SLD resolution. Equivalently, a method of d-SLD resolution can be defined following the lines of SLD.

At the end of this section we illustrate the difference in expressive power of the Horn and d-Horn restrictions with some examples.

0.4.1 Context Free Grammars as d-Horn Axioms

As Heschen and Wos report it (1974), Horn sets occur in many fields of mathematiccs such as elementary group theory, ring theory, the theory of Moufang loops, Henkin models and Boolean algebras. One may wonder about the significance of the d-Horn restriction and whether there are meaningful d-Horn theories. Our interest on the d-Horn fragment was originally motivated by the idea of representing a Context Free Grammar (CFG) as a set of FO axioms, such that production rules become d-Horn sentences. Let's first introduce the basic notions.

Briefly, any recursively enumerable language can be generated through a grammar (Hopcroft and Ullman 1979).

Definition 10 A *grammar* is a quadruple $G = (N, T, S, P)$ where N is a finite set of non-terminal symbols, T is a finite set of terminal symbols, N and T are disjoint, S is a distinguished non-terminal symbol called the start symbol, and P is a finite subset of $(N \cup T)^* \times (N \cup T)^*$. If $(L, R) \in P$ we write $L \rightarrow_G R$.

The *language generated by* G, $L(G)$, is obtained by means of the $\Rightarrow$ relation. Define $w \Rightarrow w'$ iff $w, w' \in (N \cup T)^*$, $w = xLx', w' = xRx'$ and $L \rightarrow_G R$. Then

$$L(G) = \{w \in T^* \mid S \Rightarrow^* w\}.$$

Only sentential forms over the terminal alphabet T belong to the language $L(G)$, the non-terminal symbols in N are auxiliary.

Context free grammars are characterized as those grammars G such that for all $L \rightarrow_G R$, L is a single non-terminal symbol[1].

A typical reading of a production rule like $A \rightarrow_G BC$ is that the non-terminal symbol A can can be rewritten as B followed by C and this is repeted until (presumably) we arrive to a string solely composed of terminals. This "procedural" or "incremental" reading of rewriting rules is the usual way to understand grammars. However, there is a different, more declarative reading of a grammar. A grammar is the description of a definite set of words on the alphabet of terminal symbols. Among the set of all words just some are "admitted" by the grammar, just some

[1]This feature will be of fundamental importance in our formalization of CFG through d-Horn sentences.

of them belong to the intended language, just some are "grammatical" expressions. The next step seems natural, FO (from a model theoretic point of view), is considered a language for the definition of sets. Our aim is to characterize a CFG through an FO theory obtained from its production rules. A similar approach, using modal logics, can be found in (Blackburn and Meyer-Viol 1994).

Axiomatizing CFGs. Fix a grammar $G = (N, T, S, P)$. For each non-terminal symbol A in the grammar, define a unary relation R_A over elements of T^*. For each terminal symbol $a \in T$ we define a constant a.

The axiomatization is given in the first-order language with no function symbols, and we recast the classical yuxtaposition (concatenation) function on words of T^* as a relation on T^* in the obvious way. The yuxtaposition relation $\mathrm{yux}(x_1, x_2, x)$ holds whenever $x = x_1 x_2$. This relation can be axiomatized by Horn $\cup$ d-Horn formulas by requiring associativity and functionality

$$(\forall u v w w_1 w_2 z) \begin{pmatrix} \mathrm{yux}(u, v, w_1) \\ \wedge \\ \mathrm{yux}(w_1, w, z) \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathrm{yux}(v, w, w_2) \\ \wedge \\ \mathrm{yux}(u, w_2, z) \end{pmatrix}.$$

$$(\forall x y \exists w)(\mathrm{yux}(x, y, w)) \wedge (\forall x y w w') \begin{pmatrix} \mathrm{yux}(x, y, w) & \\ \wedge & \rightarrow w = w' \\ \mathrm{yux}(x, y, w') & \end{pmatrix}.$$

We should now encode the productions. Each production rule $L \rightarrow_G R$ becomes a material implication sentence asserting the relations that hold for appropriate subwords. The following is a well known result for CFGs (Hopcroft and Ullman 1979).

Proposition 12 (Normal Form) Let $G = (N, T, S, P)$ be a context free grammar, then there exists $G' = (N', T, S, P')$ such that all the productions in P' have the form $A \rightarrow_{G'} A'A''$ or $B \rightarrow_{G'} t$ where $\{A, A', A'', B\} \subseteq N'$ and $t \in T$, and such that $L(G) = L(G')$.

We can then assume that all productions in G are already of the form $A \rightarrow A'A''$ or $B \rightarrow t$. The translation now is simple.

$$\begin{array}{ll} A \rightarrow_G A'A'' & \Rightarrow (\forall x \exists x_1 x_2)(R_A(x) \rightarrow (R_{A'}(x_1) \wedge R_{A''} \wedge \mathrm{yux}(x_1, x_2, x)) \\ B \rightarrow_G t & \Rightarrow (\forall x)(R_B(x) \rightarrow x = t). \end{array}$$

The translation provides clearly a set of Horn $\cup$ d-Horn sentences.

Example 1 Consider the context free grammar G with the production $S \rightarrow aSb \mid ab$, where S is the start symbol (the only non-terminal) and

a, b are the terminal symbols, which defines the following language

$$L(G) = \{a^i b^i \mid i \geq 1\}.$$

An equivalent grammar in normal form is $G' = (\{S, S', N_a, N_b\}, \{a, b\}, S, P')$, where P' contains the rules

$$\begin{array}{l} S \rightarrow_{G'} N_a N_b \\ S \rightarrow_{G'} N_a S' \\ S' \rightarrow_{G'} S N_b \\ N_a \rightarrow_{G'} a \\ N_b \rightarrow_{G'} b. \end{array}$$

Assume a first-order language with constant symbols $\{a, b\}$, and the unary predicate symbols $\{R_S, R_{S'}, R_{N_a}, R_{N_b}\}$. Then, the productions are translated into

$$\begin{array}{c} (\forall x \exists x_1 x_2)(R_S(x) \rightarrow (R_{N_a}(x_1) \wedge R_{N_b}(x_2) \wedge \mathrm{yux}(x_1, x_2, x))) \\ (\forall x \exists x_1 x_2)(R_S(x) \rightarrow (R_{N_a}(x_1) \wedge R_{S'}(x_2) \wedge \mathrm{yux}(x_1, x_2, x))) \\ (\forall x \exists x_1 x_2)(R_{S'}(x) \rightarrow (R_S(x_1) \wedge R_{N_b}(x_2) \wedge \mathrm{yux}(x_1, x_2, x))) \\ (\forall x)(R_{N_a}(x) \rightarrow x = a) \wedge (\forall x)(R_{N_b}(x) \rightarrow x = b). \end{array}$$

The final task is to formalize the derivation relation $\Rightarrow^*$. If a given word w belongs to the language generated by a certain grammar G, it means that *there is a* derivation for w in G, starting from the start symbol. Our formalization recasts the notion of derivability as *consistency* with the axioms for a grammar G, as follows: A word w is yielded by a grammar G with start symbol S iff the start relation on a given word w together with the axioms for G are satisfiable. Let Γ the first-order formalization of the CFG G. Then

$$S \Rightarrow^* w \text{ iff } S(w) \cup \Gamma \text{ is } \mathsf{FO} \text{ consistent}$$

or equivalently,

$$S \not\Rightarrow^* w \text{ iff } S(w) \cup \Gamma \text{ is } \mathsf{FO} \text{ inconsistent iff } \Gamma \models \neg S(w).$$

I.e., a word w does not belong to the language generated by the grammar iff the negation of the start relation over the word is derivable form the axioms. And as the formalization is performed entirely into the Horn $\cup$ d-Horn fragment, we have effective *logical* methods to check this property.

0.4.2 Illustrating d-Horn Characterization Results

This section deals with examples of FO sentences illustrating (proper) membership in the Horn and d-Horn restrictions. Let's give a concrete example to illustrate the translation function over the first-order language, the translation function over models and the preservation result for d-Horn formulae. Let φ be the characteristic axiom for connectedness, which is a d-Horn sentence.

$$\varphi = (\forall xy)(P(x,y) \vee P(y,x)) = (\forall xy)\psi(x,y)$$

(observe that φ implies reflexivity of the relation involved.) Let the index set be $I = \{1,2,3\}$ and let $\mathcal{A}_i, i \in I$ be the following models models in $\mathcal{L}$ satisfying φ:

$\mathcal{A}_1 = \langle \{a\}, R_1 \rangle$, where $R_1 = \{(a,a)\}$,
$\mathcal{A}_2 = \langle \{a,b\}, R_2 \rangle$, where $R_2 = \{(a,a),(b,b),(a,b)\}$, and
$\mathcal{A}_3 = \langle \{a,b\}, R_3 \rangle$, where $R_3 = \{(a,a),(b,b),(b,a)\}$.

Now let's turn to the translation function over φ, which yields a Horn sentence. ($t(\varphi)$ implies irreflexivity of the relation it denotes.)

$$t(\varphi) = (\forall xy)(\neg P(x,y) \vee \neg P(y,x)).$$

Let's apply the translation function over the $\mathcal{A}_i$ models:

$t(\mathcal{A}_1) = \langle \{a\}, R_1^c \rangle$, where $R_1^c = t(R_1) = \{\}$,
$t(\mathcal{A}_2) = \langle \{a,b\}, R_2^c \rangle$, where $R_2^c = t(R_2) = \{(b,a)\}$, and
$t(\mathcal{A}_3) = \langle \{a,b\}, R_3^c \rangle$, where $R_3^c = t(R_3) = \{(a,b)\}$.

We want to illustrate preservation of d-Horn sentences under dual reduce products:

$$\{i \in I : \mathcal{A}_i \models \varphi[a^1(i), \ldots, a^n(i)]\} \in D \Rightarrow \Pi_D t(\mathcal{A}_i) \models t(\varphi)[a_D^1 \ldots a_D^n].$$

Consider the proper filter $D = \{\{2,3\},\{1,2,3\}\}$ on I. $\Pi_D t(\mathcal{A}_i) = \langle \Pi_D A_i, R \rangle$ where $\Pi_D A_i = \{[(aaa)],[(aba)],[(aab)],[(abb)]\}$ and $R = \{[(aba)],[(aab)]\}$ since $\{2,3\} = \{i \in I \mid R_i^c(f^1(i), f^2(i)\}$. We can readily check that $\Pi_D(t(\mathcal{A}_i))$ satisfies $t(\varphi)$. Namely,

$$\Pi_D(t(\mathcal{A}_i)) \models (\forall xy)(\neg P(x,y) \vee \neg P(y,x)).$$

We can also use the example to see that $t(\Pi_D t(\mathcal{A}_i)) \models \varphi[a_D^1 \ \ldots a_D^n]$. Given that φ is a sentence, we can drop the assignment. $t(\Pi_D t(\mathcal{A}_i)) = t(\langle \Pi_D A_i, \{([aba],[aab])\}\rangle) = \langle \Pi_D A_i, S \rangle$, where $S = \{(x,y) \mid x,y \in \Pi_D A_i\} \setminus \{([aba],[aab])\}$. Clearly,

$$\langle \Pi_D A_i, S \rangle \models (\forall xy)(P(x,y) \vee P(y,x)).$$

We can use the example further to illustrate that the pair $([aba],[aab])$ is not in the relation R for the dual reduced product construction. By definition of relations in a dual reduced product, $\{([aba],[aab])\} \in R$ iff there exists an ultrafilter $U \supseteq D$ such that $\{i \in I : R_i(f^1(i), \ldots, f^n(i))\} \in U$. Specifically,

$$\{([aba],[aab])\} \in R \Leftrightarrow \exists U s.t. \{i \in I : R_i([aba](i),[aab](i))\} = \{1\} \in U$$

Given that gen{2} and gen{3} are the only ultrafilters extending D, and clearly $\{1\} \notin$ gen{2} and $\{1\} \notin$ gen{3}, we have $([aba], [aab]) \notin R$.

A d-Horn formula which is not Horn We know from its syntactic form that the characteristic axiom for connectedness is d-Horn. Let's prove now that there is no equivalent Horn formula. We will use the characterization result for Horn formulae that says that a formula $\varphi(x_1 \dots x_n)$ is equivalent to a Horn formula iff

$$\{i \in I : \mathcal{A}_i \models \varphi[a^1(i), \dots, a^n(i)]\} \in D \Rightarrow \Pi_D \mathcal{A}_i \models \varphi[a^1_D \dots a^n_D].$$

We can reuse models $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and the filter $D = \{\{2,3\}, \{1,2,3\}\}$ in order to show that their reduced product does not validate the formula of connectedness.

$$\Pi_D \mathcal{A}_i \not\models \psi[([aba], [aab])].$$

As $\Pi_D \mathcal{A}_i$ is not a model for φ, φ is not equivalent to a Horn formula.

d-Horn formulae are not preserved under direct products As expected, d-Horn formulae in general are not preserved under direct products. Again let's reuse the example to prove it.

$$\mathcal{A}_2 \times \mathcal{A}_3 = \langle A_2 \times A_3, R \rangle,$$

where $R = \langle$ $([aa], [aa])$, $([ab], [ab])$, $([ba], [ba])$, $([bb], [bb])$, $([aa], [ba])$, $([ab], [aa])$, $([ab], [bb])$, $([bb], [ba])$ $\rangle$. It suffices to show that there is an instance where ψ is not satisfied. For example, $\mathcal{A}_2 \times \mathcal{A}_3 \not\models \psi([aa], [bb])$.

Neither Horn nor d-Horn Consider the sentence $\varphi = (\forall xyz)(P(x,y) \wedge P(y,z) \rightarrow P(x,z) \vee P(z,x)) = (\forall xyz)\psi(x,y,z)$. We prove that φ is not equivalent to a Horn formula.

Take models $\mathcal{A}_1 = \langle \{a,b,c\}, R_1 \rangle$ such that $R_1 = \{(a,b), (b,c), (a,c)\}$ and $\mathcal{A}_2 = \langle \{a,b,c\}, R_2 \rangle$ such that $R_2 = \{(a,b), (b,c), (c,a)\}$. Clearly $\mathcal{A}_i \models \varphi$. Let's consider the trivial filter $D = \{1,2\}$. Now,

$$\Pi_D \mathcal{A}_i \not\models \psi[[(a,a)], [(b,b)], [(c,c)]].$$

Therefore φ is not equivalent to any Horn formula.

Let's see now that φ is not equivalent to a d-Horn formula. Consider two new models $\mathcal{A}_1 = \langle \{a,b,c\}, R_1 \rangle$ such that $R_1 = \{(a,b)\}$ and $\mathcal{A}_2 = \langle \{a,b,c\}, R_2 \rangle$ such that $R_2 = \{(b,c)\}$. Clearly $\mathcal{A}_i \models \varphi$. Consider again the trivial filter $D = \{1,2\}$. Now,

$$\Pi^*_D \mathcal{A}_i \not\models \psi[[(a,a)], [(b,b)], [(c,c)]].$$

Therefore φ is not equivalent to any d-Horn formula.

0.5 Conclusions and Future Work

Henschen and Wos in (1974) comment on renaming functions and Lewis in (1978) presents an algorithm to decide whether a finite set of propositional formulas can be renamed to a Horn set. Clearly our translation function $t^{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{L}$ is a renaming function: the one that complements all predicate symbols but the identity. Our work extends Henschen and Wos' by introducing translations between models and investigating new model theoretic characterizations. Besides, our framework clarifies Henschen's and Wos' remark that renaming functions do not complement the identity predicate: the identity relation can not alter its meaning. It seems straightforward to apply the ideas of this work to obtain a model theoretic characterization of other renameable Horn formulas.

We have seen that renaming functions are required to preserve satisfiability, or in other words, they carry over deduction. Following Peirce (1923), given a theory and a sentence two different sorts of logical inference can be performed, deduction and abduction, being dual forms of inference with respect to Modus Ponens. Consider the sentence $(\varphi \wedge \psi) \rightarrow \mu$. Given $(\varphi \wedge \psi)$, the sentence μ can be deduced. However, given μ, via abduction $(\varphi \wedge \psi)$ is obtained. Numerous logic oriented applications in Artificial Intelligence are based on abductive reasoning (e.g. causation, explanation, language and image interpretation). It is not difficult to realize that the translation function $t^{\mathcal{L}}$ carries through the abductive inference too. For any theory Γ and formulas φ, θ $\Gamma \cup \{\theta\} \models \varphi$ iff $t^{\mathcal{L}}(\Gamma) \cup \{t^{\mathcal{L}}(\theta)\} \models t^{\mathcal{L}}(\varphi)$ iff $t^{\mathcal{L}}(\Gamma) \cup \{t^{\mathcal{L}}(\neg\varphi)\} \models t^{\mathcal{L}}(\neg\theta)$. This last expression reveals a special behavior when φ, θ are atomic non-identity formulas, obtaining $\Gamma \cup \{\theta\} \models \varphi$ iff $t^{\mathcal{L}}(\Gamma) \cup \{\varphi\} \models \theta$. This is a curious correlation since models for Γ and models for $t^{\mathcal{L}}(\Gamma)$ might not coincide.

A number of problems remain as future work. The same dual pattern explored in this study seems applicable to other sets of FO as the positive formulas, preserved via homomorphisms. As already remarked it can be of considerable interest to obtain a simple direct proof of our d-Horn characterization result. Then, following the reverse path we have used in this work, we could provide for a simpler indirect Horn characterization result. A quite different line of research is to study the Horn and d-Horn restrictions of Modal Logic (ML), requiring the appropriate translation functions. Given that the Deduction Theorem does not hold in general for ML, it would be interesting to consider the correlation between abductive and deductive inference over theories and translated theories.

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