Interpolation and Bisimulation in Temporal Logic

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Abstract

Building on recent model theoretic results for Since-Until logics we define an adequate notion of bisimulation and establish general theorems concerning the interpolation property. Using these general results we prove that the basic SU-logic and any SU-logic whose class of frames can be defined by universal Horn formulas have interpolation. In particular, the SU-logic of branching time has interpolation, while linear time fails to have this property.

1 Introduction

For many years, modal logic (ML) was viewed as an extension of propositional logic (PL) by the addition of new modalities $\Diamond$ and $\Box$. Nowadays the picture has changed in many directions. First, modal logic is no longer seen as just an extension of PL but also as a restriction of first-order logic (FO) — when formulas are interpreted over models, or second-order logic (SO) — when formulas are considered on frames. Furthermore, $\Diamond$ and $\Box$ have lost their privileged position as a wide variety of new modalities have been introduced in the last years, witness for instance the work on Since-Until Logics [7], the Universal Modality [8], the Difference Operator [16], Counting Modalities [9], Arrow Logics [20].[1]

The source of the first change is probably Johan van Benthem [5] and his introduction of the standard translation mapping classical modal formulas to FO (or SO) formulas, and the corresponding notion of bisimulation as the equivalent to the notion of partial isomorphism known from classical model theory. With the aid of bisimulations an important number of model theoretic results for classical modal logic were derived [17, 10].

The natural next step is to try and reproduce these results for the other modal logics we mentioned. The work for SU-logics was started by Kurtonina and de Rijke in [11]. In that paper the authors provide a notion of

[1] In this article we will refer to the modal logic of $\Diamond$ and $\Box$ as classical modal logic.

bisimulation for SU-logics with the standard relational semantics, and they establish a characterization result and a number of separation and preservation results. It turns out, though, that the standard relational semantics is not completely adequate (witness the absence of canonical models and of an appropriate algebraic counterpart). Bellisima and Cittadini [4] have introduced a new semantics with a dual algebraic construction, and a Stone-like representation theorem and a canonicity result have been proved. The present paper is based in this new semantics. We start by defining the appropriate notion of bisimulation and then focus on the use of bisimulations to prove general results about the interpolation property, leaving other model theoretic considerations for further research. Recent results concerning interpolation by means of bisimulations can be found in [2, 14] for classical modal logic and in [3, 19] for infinitary classical modal logic. In contrast, surprisingly little is known about interpolation (and metalogical properties in general) for SU-logics.

In what follows, we assume basic knowledge of modal logic. In Section 2 we summarize the required definitions and results about the interpolation property. In Section 3 we recall two general results about interpolation and failure of interpolation for classical modal logic, and we extend these results to SU-logics in Section 4. In Section 5 we apply these two theorems to special cases and establish interpolation for the basic SU-logic K_{SU} and for any SU-logic whose class of frames can be defined by universal Horn clauses in FO. An important case covered by this result is branching time (the class of frames where the accessibility relation is a partial order). We also prove that linear time fails to have interpolation. Finally, in Section 6 we comment on the results obtained and discuss further directions of research.

2 The Interpolation Property

The interpolation property (IP) is an important metalogical notion. Originally, IP was a syntactic property of a given deductive system. In that framework, a deductive system has the IP if whenever $A \vdash B$ then there exists a formula C in the common language of A and B such that $A \vdash C$ and $C \vdash B$. As a syntactic property, the IP is a sign of a well behaved deductive system. It ammounts to the fact that when proving B from A, intermediate lemmas can be restricted only to the common language. Obviously, once a soundness and completeness result for a given logic is obtained, the IP can be also stablished by semantic means and this is perhaps the standard approach nowadays [6].

Besides its wide use the area of automatic theorem proving (i.e. [1, 18]), the interpolation property has turned to be interesting in fields like, for exaple, software engeneering where it can be used to proved certain modularity properties of the specification of a system [12, 15].

Our approach to the IP is purely semantical. For K a class of models (say of FO), let $\models_{\mathsf{K}}$ be the standard semantic consequence relation for K: for $\Phi \cup \{\psi\}$ a set of sentences, $\Phi \models_{\mathsf{K}} \psi$ iff all models of Φ in K are also models of ψ. As usual, for $\Phi = \{\varphi\}$ we use $\varphi \models_{\mathsf{K}} \psi$, and $\models_{\mathsf{K}} \psi$ instead of $\emptyset \models_{\mathsf{K}} \psi$.

The following definitions of interpolation can be found in the literature. Let $I\!P(\varphi)$ be the set of atomic symbols occurring in φ (proposition variables in ML, relation symbols in FO) and let $\mathcal{L} = \mathrm{Th}(\mathsf{K})$ be the theory of a class of models K.

AIP $\mathcal{L}$ has the *Arrow Interpolation Property* (AIP) if, whenever $\models_{\mathsf{K}} \varphi \to \psi$, there exists a formula θ such that $\models_{\mathsf{K}} \varphi \to \theta$, $\models_{\mathsf{K}} \theta \to \psi$ and $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

TIP $\mathcal{L}$ has the *Turnstile Interpolation Property* (TIP) if, whenever $\varphi \models_{\mathsf{K}} \psi$, there exists a formula θ such that $\varphi \models_{\mathsf{K}} \theta$, $\theta \models_{\mathsf{K}} \psi$ and $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

SIP $\mathcal{L}$ has the *Splitting Interpolation Property* (SIP) if, whenever $\varphi_0 \wedge \varphi_1 \models_{\mathsf{K}} \psi$, there exists a formula θ such that $\varphi_0 \models_{\mathsf{K}} \theta$, $\varphi_1 \wedge \theta \models_{\mathsf{K}} \psi$ and $I\!P(\theta) \subseteq I\!P(\varphi_0) \cap (I\!P(\varphi_1) \cup I\!P(\psi))$.

For FO the above definitions are all equivalent but in general this is not the case (depending on both compactness and the availability of a deduction theorem in the logic). The meaning of TIP and SIP in modal logic depends on the way we define the consequence relation $\varphi \models_{\mathsf{K}} \psi$. There are two options: a local and a global one (cf., [5] or [14] for a discussion of their relative merits.) Let K be a class of frames.

the *local consequence relation* $\Phi \models^{loc}_{\mathsf{K}} \psi$ holds if for every $\mathcal{F} \in \mathsf{K}$, every valuation v and every world w in F, $\langle \mathcal{F}, v\rangle, w \Vdash \Gamma$ implies $\langle \mathcal{F}, v\rangle, w \Vdash \psi$,

the *global consequence relation* $\Gamma \models^{glo}_{\mathsf{K}} \psi$ holds if for every $\mathcal{F} \in \mathsf{K}$ and every valuation v, $\langle \mathcal{F}, v\rangle \models \Gamma$ implies $\langle \mathcal{F}, v\rangle \models \psi$.

The global relation is the one familiar from first-order logic, but it is always defined for Φ a set of *sentences* (if they are formulas, the universal closure is considered.) If we view the world w as an assignment, then for sentences as premises, the two notions are equivalent. Indeed, when Φ is a set of formulas —and they are treated as formulas— the local definition becomes the more interesting one (cf. the definition just before Proposition 2.3.6 in [6]).

Proposition 2.1
(i) With the local consequence relation, AIP, TIP and SIP are all equivalent.
(ii) If $\models^{glo}_{\mathsf{K}}$ *is compact, then AIP implies TIP, and TIP and SIP are equivalent.*

Notice that, by Proposition 2.1, if a logic is compact (which is always the case if the class of frames is elementary), a proof of AIP implies that all the other kinds of interpolation also hold. Conversely, disproving SIP (or TIP) implies the failure of all of them. In the rest of the article $\models$ refers always to the global consequence relation.

3 Interpolation in Classical Modal Logic

Important general results concerning interpolation for standard modal logics are known, witness [13]. These results are a byproduct of the strong connections between the interpolation property and the algebraic property of amalgamation. In this paper we will discuss two recent results providing, respectively, a method to prove AIP and a method to disprove SIP. First, the following notions should be introduced

Definition 3.1 Let $\mathcal{F}^c_{\mathcal{L}}$ be the canonical frame for the logic $\mathcal{L}$. Then $\mathcal{L}$ is *canonical* if $\mathcal{F}^c_{\mathcal{L}} \models \mathcal{L}$.

Definition 3.2 Let $\mathcal{G} = \langle G, R_{\mathcal{G}} \rangle$ and $\mathcal{H} = \langle H, R_{\mathcal{H}} \rangle$ be two frames. Let $B \subseteq G \times H$ be nonempty.

1. We say that B is a *bisimulation* between $\mathcal{G}$ and $\mathcal{H}$ if whenever gBh and $gR_{\mathcal{G}}g'$, then there exists $h' \in H$ such that $hR_{\mathcal{H}}h'$ and $g'Bh'$, and similarly in the other direction. If gBh holds we will call g and h *bisimilar*.

2. If B is a total function f, then it is called a *zigzag morphism*. If f is also surjective we use the notation $\mathcal{G} \stackrel{f}{\twoheadrightarrow} \mathcal{H}$, and call $\mathcal{H}$ the *zigzag morphic image* of $\mathcal{G}$ by f.

3. The notions of bisimulation and zigzag morphism can also be defined for models $\mathfrak{M}_{\mathcal{G}} = \langle \mathcal{G}, v_{\mathcal{G}} \rangle$ and $\mathfrak{M}_{\mathcal{H}} = \langle \mathcal{H}, v_{\mathcal{H}} \rangle$, relative to a given set of

propositional variables $I\!P$ by adding the condition: if gBh then for all $p_i \in I\!P$, $\mathfrak{M}_{\mathcal{G}}, g \Vdash p_i$ iff $\mathfrak{M}_{\mathcal{H}}, h \Vdash p_i$. We will say in this case that B is a *$I\!P$-bisimulation* or a *$I\!P$-zigzag morphism.*

Definition 3.3 Let $\mathcal{F}$ and $\mathcal{G}$ be two frames. Let $\mathcal{H}$ be a submodel of the direct product $\mathcal{F} \times \mathcal{G}$. $\mathcal{H}$ is called a *zigzag product* of $\mathcal{F}$ and $\mathcal{G}$ if the projections are surjective zigzag morphisms. We say that a class K of frames is *closed under zigzag products* if every zigzag product of two frames in K is also in K.

Theorem 3.4 ([14]) *Let $\mathcal{L}$ be a canonical classical modal logic. If the class of frames of $\mathcal{L}$ is closed under zigzag products, then $\mathcal{L}$ has the AIP.*

Theorem 3.5 ([2]) *Let K be a class of frames, and $\mathcal{L}_{\mathsf{K}}$ be the classical modal logic of K.*

SIP fails in $\mathcal{L}_{\mathsf{K}}$ if there are finite frames $\mathcal{G}, \mathcal{H} \in \mathsf{K}$, a finite frame $\mathcal{F}$ and surjective zigzag morphisms m, n such that $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$, $\mathcal{F}$ is generated by one point f, every m-pre-image of f in G generates $\mathcal{G}$, and similarly for $\mathcal{H}$, and there is no frame $\mathcal{J} \in \mathsf{K}$ with commuting surjective zigzag morphisms g and h from $\mathcal{J}$ onto $\mathcal{G}$ and $\mathcal{H}$ (i.e. $\mathcal{G} \overset{g}{\twoheadrightarrow} \mathcal{J} \overset{h}{\twoheadleftarrow} \mathcal{H}$ and $m \circ g = n \circ h$.)

Moreover, an explicit counterexample for SIP can be algorithmically constructed from the frames and functions $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$.

Theorem 3.4 provides a tool to prove interpolation. We need only verify that the class of frames of our logic is closed under a given model theoretic construction. As a corollary all (canonical) classical modal logics whose class of frames can be defined by universal Horn sentences have interpolation [14], as these formulas are preserved under zigzag products. Theorem 3.5 shows how in some special cases, failure of closure under zigzag products produces failure of interpolation. Using this method, failure of interpolation for finite variable fragments of FO, the difference modality, Humbersone's innacessibility operator, and different versions of product logics and union logics has been proved (or re-proved) in [2]. In the present paper we extend these results to SU-logics.

4 General Results for Since-Until Logics

What do we need to extend the main results of Section 3 to SU-logics? We need canonical frames and bisimulations. So far neither was available. In [4] Bellisima and Cittadini introduce a new semantics for SU-logics and prove it to be strongly adequate by providing a Stone-like duality theorem between the "new" general frames (called e-frames) and algebras with operators u and s defined in the obvious way. We will not discuss the algebraic counterpart in this paper as we are mainly concerned with a modal aproach. But our results are based on their semantics and their results concerning the existence of canonical models.

Definition 4.1 ([4]) An *e-frame* is a Kripke frame $\langle W, R\rangle$ together with a function β from R into $\mathcal{P}(\mathcal{P}(W))$ such that for every x and y with xRy we have $\beta(x,y) \neq \emptyset$ and, if $Z \in \beta(x,y)$, then $Z \subseteq \{z \mid xRzRy\}$. $\beta(x,y)$ can be thought of as the sets of relevant points situated between x and y.

An *e-model* is an e-frame together with a valuation. The truth definition for e-models is standard for propositional variables and Boolean connectives. Furthermore, $x \models U(\varphi,\psi)$ if there exists a point y such that xRy and $y \models \varphi$ and there exists $Z \in \beta(x,y)$ such that $z \models \psi$ for each $z \in Z$, and analogously for S.

For this semantics the appropriate canonical models and frames are defined as follows.

Definition 4.2 ([4]) Given any US-logic $\mathcal{L}$ we define its *canonical model* $\mathfrak{M}_{\mathcal{L}}$ as the e-model $\langle W_{\mathcal{L}}, R_{\mathcal{L}}, \beta_{\mathcal{L}}, v_{\mathcal{L}}\rangle$ where:

1. $W_{\mathcal{L}}$ is the set of all maximal consistent extensions of $\mathcal{L}$.
2. $xR_{\mathcal{L}}y$ iff $U(\varphi,\top)$ belongs to x for any $\varphi \in y$.
3. Let $xR_{\mathcal{L}}y$ and $Z \subseteq \{z \mid xR_{\mathcal{L}}xR_{\mathcal{L}}y\}$, then $Z \in \beta_{\mathcal{L}}(x,y)$ iff for any φ, ψ such taht $\varphi \in y$ and, for each $z \in Z$, $\psi \in z$, it holds $U(\varphi,\psi) \in x$.
4. $v_{\mathcal{L}}(p) = \{x \in W_{\mathcal{L}} \mid p \in x\}$, for any variable p.

We now introduce the appropriate notions of e-zigzag morphism and e-bisimulations.

Definition 4.3 Let $\mathcal{G} = \langle G, R_{\mathcal{G}}, \beta_{\mathcal{G}}\rangle$ and $\mathcal{H} = \langle H, R_{\mathcal{H}}, \beta_{\mathcal{G}}\rangle$ be two frames. Let $B \subseteq G \times H$ be nonempty.

1. We say that B is an *e-bisimulation* between $\mathcal{G}$ and $\mathcal{H}$ if the following clauses hold:

   (a) If gBh and $gR_{\mathcal{G}}g'$, then there exists $h' \in H$ such that $hR_{\mathcal{H}}h'$ and $g'Bh'$. Furthermore for all $Z \in \beta_{\mathcal{G}}(g,g')$ there exists $Z' \in \beta_{\mathcal{H}}(h,h')$

such that for all $h'' \in Z'$ there is $g'' \in Z$ and $g''Bh''$.
(b) Clause (a) with $R^c_{\mathcal{G}}$ and $R^c_{\mathcal{H}}$ instead of $R_{\mathcal{G}}$ and $R_{\mathcal{H}}$.
(c) Clauses (a) and (b) but going from $\mathcal{H}$ to $\mathcal{G}$.

If gBh holds we will call g and h *e-bisimilar*.

2. If B is a total function f, then it is called an *e-zigzag morphism*. If f is also surjective we use the notation $\mathcal{G} \overset{B}{\twoheadrightarrow} \mathcal{H}$), and call $\mathcal{H}$ the *e-zigzag morphic image* of $\mathcal{G}$ by f.

3. The notions of e-bisimulation and e-zigzag morphism can also be defined for models $\mathfrak{M}_{\mathcal{G}} = \langle \mathcal{G}, v_{\mathcal{G}} \rangle$ and $\mathfrak{M}_{\mathcal{H}} = \langle \mathcal{H}, v_{\mathcal{H}} \rangle$, relative to a given set of propositional variables $I\!P$ by adding the condition: if gBh then for all $p_i \in I\!P$, $\mathfrak{M}_{\mathcal{G}}, g \Vdash p_i$ iff $\mathfrak{M}_{\mathcal{H}}, h \Vdash p_i$. We will say in this case that it is a *$I\!P$-e-bisimulation* or a *$I\!P$-e-zigzag morphism*.

E-bisimulations are defined to make the following proposition true.

Proposition 4.4 *Let $\mathfrak{M}_{\mathcal{G}}$ and $\mathfrak{M}_{\mathcal{H}}$ be two e-models and B a $I\!P$-e-bisimulation between them. Then for every SU-formula φ constructed from variables in $I\!P$, gBh implies $g \Vdash \varphi \Leftrightarrow h \Vdash \varphi$.*

Once the correct definition of bisimulation is obtained and we have a "nice" semantics, Theorems 3.4 and 3.5 can be re-proved for SU-logic. The only important step is to check the needed lemmas in [14] and [2].

Notation. Let $\mathcal{L}_{SU}$ be the set of all formulas in a SU-logic. For $\varphi \in \mathcal{L}_{SU}$, let $\mathcal{L}_{\varphi}$ be the restriction of $\mathcal{L}_{SU}$ to the propositional symbols in φ. $\mathfrak{M}_{\varphi} = \langle W_{\varphi}, R_{\varphi}, \beta_{\varphi}, v_{\varphi} \rangle$ is the canonical model over $\mathcal{L}_{\varphi}$.

For the positive characterization the following result is vital:

Lemma 4.5 *Let $\varphi, \psi \in \mathcal{L}_{SU}$. Let $\mathfrak{M}_{\varphi}$, $\mathfrak{M}_{\psi}$ be the corresponding canonical models. Then*

$$B = \{(w, v) \in W_{\varphi} \times W_{\psi} \mid \mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi} \cap w = \mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi} \cap v\}$$

is a total, surjective $(\mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi})$-e-bisimulation between $\mathfrak{M}_{\varphi}$ and $\mathfrak{M}_{\psi}$.

Proof. B is a relation on $W_{\varphi} \times W_{\psi}$ by definition. That B is total (surjective) can be proved from the fact that if $w \in W_{\varphi}$ ($v \in W_{\psi}$) then $\mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi} \cap w$ ($\mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi} \cap v$) is $(\mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi})$-consistent and hence can be extended to a set in W_{ψ} (W_{φ}). To prove that B satisfies the conditions in the definition of e-bisimulation it is enough to notice the following.
Let $\mathfrak{M}_{\varphi,\psi} = \langle W_{\varphi,\psi}, R_{\varphi,\psi}, \beta_{\varphi,\psi}, v_{\varphi,\psi} \rangle$ be the canonical model on $\mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi}$.

(i.) Let $x, y \in W_\varphi$ and $Z \in \beta_\varphi(x,y)$, then $Z'' = \{z \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z \in Z\} \in \beta_{\varphi,\psi}(x \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi, y \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi)$.

This is simple. Let $x'' = x \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ and $y'' = y \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ then $x'', y'' \in W_{\mathcal{L}_\varphi \cap \mathcal{L}_\psi}$. Also $x'' R_{\mathcal{L}_\varphi \cap \mathcal{L}_\psi} y''$. Similarly $Z'' \subseteq \{z'' \mid x'' R_{\mathcal{L}_\varphi \cap \mathcal{L}_\psi} z'' R_{\mathcal{L}_\varphi \cap \mathcal{L}_\psi} y''\}$. And finally for any $\theta_1, \theta_2 \in \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ such that $\theta_1 \in y''$, if for all $z'' \in Z''$ $\theta_2 \in z''$ then $U(\theta_1, \theta_2) \in x''$ as all formulas are included in $\mathcal{L}_\varphi$ and Z satisfied the condition.

(ii.) Let $x'', y'' \in W_{\varphi,\psi}$ and $Z'' \in \beta_{\varphi,\psi}(x'', y'')$, then for all $x', y' \in W_\psi$ and $Z' \in \beta_\psi(x', y')$ such that $x' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = x''$, $y' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = y''$ and $\{z' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z' \in Z'\} = Z''$.

INCOMPLETE HERE!!!

Now suppose $Z \in \beta(g, g')$ we have to prove that there is $Z' \in \beta(h, h')$ such that for all $h'' \in Z'$ there is $g'' \in Z$ and $g'' B h''$.

Define $Z'' = \{z \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z \in Z\}$. Then by (i.) $Z'' \in \beta_{\mathcal{L}_\varphi \cap \mathcal{L}_\psi}(g \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi, g' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi)$. By (ii.) now, as $h \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = g \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ and $h' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = g' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ and for Z'' define as above, there is $Z' \in \beta(h, h')$ such that $Z'' = \{z \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z \in Z'\}$. Then if $h'' \in Z'$, $h'' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \in Z'$ and there is $g'' \in Z$ such that $g'' B h''$. QED

Once this is proved the main argument in [14] establishes the theorem.

Theorem 4.6 *Let $\mathcal{L}$ be a canonical SU-modal logic. If the class of frames of $\mathcal{L}$ is closed under e-zigzag products, then $\mathcal{L}$ has the arrow interpolation property.*

Proof. The outline of the proof in [14] is as follows.

Reason by contraposition. Suppose there is no interpolant for $\varphi \to \psi$. We will prove that $\varphi \wedge \neg\psi$ is satisfiable.

Define B as in Lemma **??**. We claim that there is $(w, v) \in B$ such that $\mathfrak{M}_\varphi, w \Vdash \varphi$ and $\mathfrak{M}_\psi, v \Vdash \neg\psi$. Define $\{\theta \in \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid \ \models \varphi \to \theta\} \cup \{\neg\theta \mid Lint\} \models \theta \to \psi$. Since there is no interpolant the set is consistent and can be extended to an element u of $W_{\varphi,\psi}$. $u \cup \{\varphi\}$ and $u \cup \{\neg\psi\}$ are also consistent and can be extended to elements $w \in W_\varphi$ and $v \in W_\psi$. This fact, plus Lemma **??**, plus the fact that the class of frames is closed under e-zigzag producs produces the model we needed. QED

For the equivalent to Theorem 3.5 what is needed is another result involving e-bisimulations. Now we should be able to describe finite e-frames up to e-bisimulation.

Lemma 4.7 *Let $\mathcal{F} = \langle F, R_{\mathcal{F}}, \beta_{\mathcal{F}} \rangle$ be a finite frame generated by f_1 and let $|F| = n$. Let $\mathfrak{M}_{\mathcal{F}} = \langle \mathcal{F}, v_{\mathcal{F}} \rangle$ be a model such that $v_{\mathcal{F}}(p_i) = \{f_i\}$ for $p_1, \ldots, p_n$. Then there is an SU-formula $\Sigma_{\mathcal{F}}$ such that for any model $\mathfrak{M}_{\mathcal{G}} = \langle G, R_{\mathcal{G}}, \beta_{\mathcal{G}} \rangle$ with $\mathfrak{M}_{\mathcal{G}} \models \Sigma_{\mathcal{F}}$ and $\mathfrak{M}_{\mathcal{G}}, g \Vdash p_1$ for some $g \in G$, the relation $B \subseteq G \times F$ defined as*

gBf iff g and f agree in the truth value assigned to $\{p_1, \ldots, p_n\}$

is a surjective $\{p_1, \ldots, p_n\}$-e-zigzag morphism from $\mathfrak{M}_{\mathcal{G}}$ and $\mathfrak{M}_{\mathcal{F}}$.

Proof. Define $\Sigma_{\mathcal{F}}$ as the conjunction of

$$
\begin{aligned}
A_1 &= \textstyle\bigvee_{1 \le k \le n} p_k, \\
A_2 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{\neg p_l \mid k \neq l\}) \\
A_3 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{\neg F p_l \mid \text{not } f_k R_{\mathcal{F}} f_l\}) \\
A_4 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{U(p_l, \bigvee_I p_i) \mid f_k R_{\mathcal{F}} f_l \\
&\qquad \text{and } I = \{i \mid Z \in \beta_{\mathcal{F}}(f_k, f_l), f_i \in Z\}\}) \\
A_5 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{\neg U(p_l, \bigvee_I p_i) \mid f_k R_{\mathcal{F}} f_l \\
&\qquad \text{and } I = \{i \mid Z \subseteq F, f_i \in Z, (\forall Z' \in \beta_{\mathcal{F}}(f_k, f_l))(Z' \not\subseteq Z)\}\}) \\
A_6 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{\neg P p_l \mid \text{not } f_l R_{\mathcal{F}} f_k\}) \\
A_7 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{S(p_l, \bigvee_I p_i) \mid f_l R_{\mathcal{F}} f_k \\
&\qquad \text{and } I = \{i \mid Z \in \beta_{\mathcal{F}}(f_l, f_k), f_i \in Z\}\}) \\
A_8 &= \textstyle\bigwedge_{1 \le k \le n} (p_k \to \bigwedge\{\neg S(p_l, \bigvee_I p_i) \mid f_l R_{\mathcal{F}} f_k \\
&\qquad \text{and } I = \{i \mid Z \subseteq F, f_i \in Z, (\forall Z' \in \beta_{\mathcal{F}}(f_l, f_k))(Z' \not\subseteq Z)\}\})
\end{aligned}
$$

By using $\Sigma_{\mathcal{F}}$, it is possible to prove that B is a surjective $\{p_1, \ldots, p_n\}$-e-zigzag morphism. We prove only the conditions for e-bisimulations, the others properties being simpler.

Assume gBf and $gR_{\mathcal{G}}g'$ to prove there exists f' such that $fR_{\mathcal{F}}f'$. Let $g \models p_i$, $g' \models p_j$, $i, j \in n$ as follows from A_1 and A_2. Hence $g \models Fp_j$, but $g \models \neg F p_l$ whenever not $f_i R_{\mathcal{F}} f_j$, so $f_i R_{\mathcal{F}} f_j$. Furthermore, by definition of B, $g' B f_j$ (*).

Let now $Z \in \beta_{\mathcal{G}}(g, g')$, hence by definition $Z \subseteq \{z \mid gR_{\mathcal{G}}zR_{\mathcal{G}}g'\}$. We want to prove that there is $Z' \in \beta_{\mathcal{F}}(f_i, f_j)$ such that for all $f'' \in Z'$ there is $g'' \in Z$ and $g''Bh''$. Again from $g \models p_i$ we obtain by A_3 and A_5 that $g \models U(p_j, \bigvee\{p_l \mid f_i R_{\mathcal{F}} w_l R_{\mathcal{F}}\} w_j)$ and $g \models \neg U(p_j, \bigvee\{p_l \mid \text{not } f_i R_{\mathcal{F}} w_l R_{\mathcal{F}}\} w_j)$. Take $z \in Z$ then

$gR_{\mathcal{G}}z$, hence reasoning as in (*) there is f' such that $f_i R_{\mathcal{F}} f'$ and zBf'.

$gR_{\mathcal{G}}g'$, hence reasoning as in (*) but using A_7 there is f'' such that $f'' R_{\mathcal{F}} f_j$ and zBf''.

By functionality of B, $f' = f''$. Define then $Z' = \{f\} zBf$ and $z \in Z$. Now if for some $Z'' \subseteq Z'$ we prove $Z'' \in \beta_{\mathcal{F}}(f_i, f_j)$ we are done.

Suppose not, $Z'' \notin \beta_{\mathcal{F}}(f_i, f_j)$ for all $Z'' \subseteq Z'$. At the same time $g \models U(p_j, \bigvee\{p_k \mid f_k \in Z'\})$ as $Z \in \beta_{\mathcal{G}}(g, g')$, $gR_{\mathcal{G}}g'$ and for all $g'' \in Z$, $g'' \models \bigvee\{p_k \mid f_k \in Z'\}$.

But $g \models \bigwedge\{\neg U(p_j, \bigvee_L p_l) \mid L = \{l \mid Z \subseteq F, f_l \in Z,$ for all $Z', Z' \in \beta_{\mathcal{F}}(f_j, f_i) \Rightarrow Z' \subseteq Z\}\}$.

Now $g \models A_3$ hence $g \models \bigwedge\{p_j, \bigvee_L p_l \mid L = \{l \mid Z \in \beta_{\mathcal{F}}(f_i, f_j), f_i \in Z\}\}$.

Fix $Z \in \beta_{\mathcal{F}}(f_i, f_j)$, therefore there exists g', $gR_{\mathcal{F}}g'$ and $g' \models p_j$. Hence there is $Z' \in \beta_{\mathcal{G}}(g, g')$ such that for all $z \in Z'$, $z \models \bigvee\{p_l \mid f_l \in Z\}$. Hence for all $z \in Z'$, exists $f_i \in Z$ such that zBf_i. QED

Theorem 4.8 *Let* K *be a class of e-frames and let* $\mathcal{L}_{\mathsf{K}}$ *be the SU-logic of* K.

SIP fails in $\mathcal{L}_{\mathsf{K}}$ *if there are finite frames* $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *and surjective zigzag SU-morphisms* m, n *such that* $\mathcal{G} \stackrel{m}{\twoheadrightarrow} \mathcal{F} \stackrel{n}{\twoheadleftarrow} \mathcal{H}$, $\mathcal{F}$ *is generated by one point* w, *every* m*-pre-image of* w *in* G *generates* $\mathcal{G}$, *and similar for* $\mathcal{H}$, *and there is no frame* $\mathcal{J} \in \mathsf{K}$ *with commuting surjective zigzag SU-morphisms* g *and* h *from* $\mathcal{J}$ *onto* $\mathcal{G}$ *and* $\mathcal{H}$ *(i.e.* $\mathcal{G} \stackrel{m}{\twoheadleftarrow} \mathcal{J} \stackrel{n}{\twoheadrightarrow} \mathcal{H}$.*)*

Moreover, an explicit counterexample for SIP can be algorithmically constructed from the frames and functions $\mathcal{G} \stackrel{m}{\twoheadrightarrow} \mathcal{F} \stackrel{n}{\twoheadleftarrow} \mathcal{H}$.

PROOF. We give the outline of the proof in [2] for completeness. Assume $\mathcal{G} \stackrel{m}{\twoheadrightarrow} \mathcal{F} \stackrel{n}{\twoheadleftarrow} \mathcal{H}$. Obtain models from the frames by providing a valuation which assigns a unique propositional symbol to the elements of the doman. Use disjoint vocabularies $\{f_1, \ldots, f_{|F|}\}$, $\{g_1, \ldots, g_{|G|}\}$ and $\{h_1, \ldots, h_{|H|}\}$. Assume f_1 is the propositional symbol true at the world generating $\mathcal{F}$. Now the formulas

$$\Gamma_m = \bigwedge_{1 \leq i \leq |F|}(f_i \leftrightarrow \bigvee\{g_j \mid m(w_j) = w_i\})$$
$$\Gamma_n = \bigwedge_{1 \leq i \leq |F|}(f_i \leftrightarrow \bigvee\{h_j \mid n(w_j) = w_i\}).$$

can be used to describe the functions m and n.

Now the following can be stablished:

(1)$(\Sigma_{\mathcal{G}} \wedge \Gamma_m) \wedge (\Sigma_{\mathcal{H}} \wedge \Gamma_n) \models \neg f_1$,

(2)there is no splitting interpolant.

The proof of the two lines above proceeds by contradiccion. Assuming the negation of (1) forces $\langle \mathcal{G}, v_{\mathcal{H}} \rangle$ to satisfy $\neg f_1$ and at the same time there is a world in H which is mapped by n to f_1. Assuming the negation of (2) let us create in K a frame $\mathcal{J}$ with commuting surjective e-zigzag morphisms onto $\mathcal{G}$ and $\mathcal{H}$ contradicting hypothesis. QED

As a final remark we notice that once a good definition of bisimulation for the logic is obtained, the results above are easily deduced. We conjecture that this is an instance of a more general fact. As we discussed in the introduction, the notion of bisimulation is one of the key semantic tools in modal logic. It seems that once the appropiate definition for a given logic is found, a number of results follows "for free." Our ongoing research is to explore this phenomenom in greater detail.

5 Applications

In this section we discuss how the general results we just proved we can put to work. We only mention a couple of specific instances where Theorems 4.6 and 4.8 apply, and further research is needed here.

Positive Results. The first positive result is immidiate. The class of all e-frames is canonical [4] and trivially closed under e-zigzag-producs. Hence

Proposition 5.1 *The basic SU-logic* K_{SU} *has AIP, TIP and SIP.*

Next, the result concerning universal Horn formulas mentioned in [14, Corollary B.4.] transfers.

Proposition 5.2 *Let* $\mathcal{L}$ *be a canonical SU-logic. If* $\mathsf{Fr}_{\mathcal{L}}$ *can be defined by universal Horn sentences, then* $\mathcal{L}$ *has the AIP, TIP and SIP.*

In the temporal interpretation of Since-Until, an important class of frames which is covered by the above corollary is the class of frames where R is a partial order.[2]

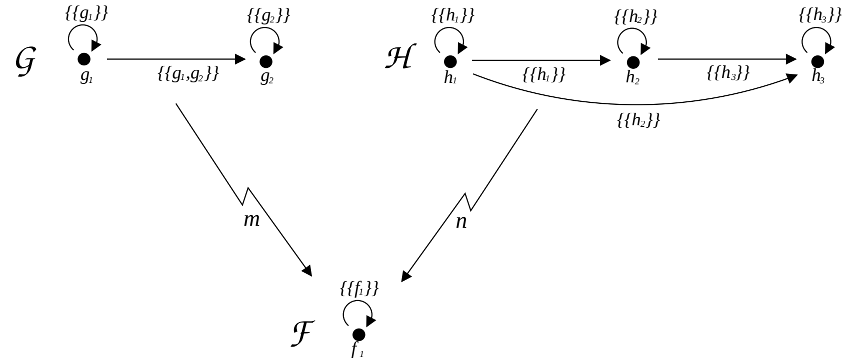
Proposition 5.3 *Let* K_{bran} *be the SU-logic of the class of e-frames where the accessibility relation is a partial order. Then AIP, TIP and SIP hold for* K_{bran}.

Negative Results. We will only exemplify Theorem 4.8 for one case: linear time. By adding the condition of totality for the accessibility relation, i.e., $(\forall x, y)(xRy \vee yRx)$, we end up outside the universal Horn fragment. We can actually prove that the SU-logic of this class of frames does not have interpolation.

Proposition 5.4 *Let* K_{lin} *be the SU-logic of the class of e-frames where the accessibility relation is a linear order. Then AIP, TIP and SIP fail for* K_{lin}.

[2] I.e., transitive, antisymetric and reflexive. It is immediate to check that these conditions are universal Horn.

PROOF. To use Theorem 4.8 we should provide three finite frames $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{F}$. We propose the following



The sets of sets labeling the accessibility relations define the β function for the e-frames. It is not difficult to check that the e-frames satisfy the conditions in Theorem 4.8. Also, the functions m and n that map all elements to f_1 are surjective e-zigzag morphisms. We will now prove that no e-frame $\mathcal{J}$ exists in K_{lin} with surjective e-zigzag morphism g and h onto $\mathcal{G}$ and $\mathcal{H}$ respectively.

When we say that $\langle g_k, h_l\rangle$ is an element of $\mathcal{J}$, we meaning that there is an element $j \in J$ such that $g(j) = g_k$ and $h(j) = h_l$. As g, h should be surjective, $\langle g_1, h_i\rangle \in J$. By the zigzag conditions it follows that $\langle g_1, h_i\rangle R_{\mathcal{J}} \langle g_2, h_j\rangle$. To satisfy the condition on $\beta_{\mathcal{J}}$, it should be the case that $h_i = h_j$. Suppose $i \in \{1, 2\}$; then $\langle g_2, h_i\rangle R_{\mathcal{J}} \langle g_k, h_{i+1}\rangle$. By definition of $R_{\mathcal{G}}$, $k = 2$. But now, by transitivity, $\langle g_1, h_i\rangle R_{\mathcal{J}} \langle g_2, h_{i+1}\rangle$ and the condition for $\beta_{\mathcal{J}}$ for this pair cannot be fulfilled. In the case where $i = 3$, we find that $\langle g_k, h_{i-1}\rangle R_{\mathcal{J}} \langle g_1, h_i\rangle$, and we reason similarly. QED

6 Conclusions

In this article we have focused on the use of bisimulation for SU-logics in the context of interpolation. We have proved that general (positive and negative) results hold, similar to those established for classical modal logic [2, 14]. By means of the two main theorems presented in Section 4, interpolation for branching time has been proved, while we have shown that linear time fails to have this property.

We conjecture that a number of similar results can be established for other modal logics which are based only on the existence of appropriate notions of bisimulation and canonical frame, as well as the availability of certain model theoretic techniques. Our ongoing research is to investigate the general pattern behind this phenomenon. We plan to investigate characterization and preservation results for SU-logics. Also, given that a very similar notion of bisimulation was given for Interpretability Logic in [21], our results may also extend in that direction.

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