Coinductive Models and Normal Forms for Modal Logics

(or How we Learned to Stop Worrying and Love Coinduction)

Carlos Areces

INRIA Nancy Grand Est, Nancy, France

Daniel Gorín

Departamento de Computación, Universidad de Buenos Aires, Buenos Aires, Argentina

Abstract

We present a coinductive definition of models for modal logics and show that it provides a homogeneous framework in which it is possible to include different modal languages ranging from classical modalities to operators from hybrid and memory logics. Moreover, results that had to be proved separately for each different language –but whose proofs were known to be mere routine– now can be proved in a general way. We show, for example, that we can have a unique definition of bisimulation for all these languages, and prove a single invariance-under-bisimulation theorem.

We then use the new framework to investigate normal forms for modal logics. The normal form we introduce may have a smaller modal depth than the original formula, and it is inspired by global modalities like the universal modality and the satisfiability operator from hybrid logics. These modalities can be *extracted* from under the scope of other operators. We provide a general definition of *extractable modalities* and show how to compute extracted normal forms. As it is the case with other classical normal forms –e.g., the conjunctive normal form of propositional logic– the extracted normal form of a formula can be exponentially bigger than the original formula, if we require the two formulas to be equivalent. If we only require equi-satisfiability, then every modal formula has an extracted normal form which is only polynomially bigger than the original formula, and it can be computed in polynomial time.

Keywords: modal logics, hybrid logics, normal forms, modal depth

1 Normal Forms and Why Coinductive Models are Nice

Finding appropriate normal forms can be extremely useful, for different reasons [3]. For example, in mathematics we can use Gauss algorithm to easily find the rank of a matrix (we can say that the normal form obtained via the Gauss algorithm provides information about matrices not directly derivable from arbitrary forms); and we can represent polynomials as vectors to simplify operations (that is, the normal form provides more efficient computation). Normal forms can also be used as representatives of a given class. In lambda calculus, for example, beta reduced lambda terms can be used as *values*, or *representatives* of lambda terms classes.

In automated deduction, transformation to normal form usually serves to simplify the inference calculus required. For example, if we already have a propositional calculus that handles $\vee$ and $\neg$ we can eliminate classical implication using the equivalence $(A \to B) \equiv (\neg A \vee B)$ and we don't need any further inference machinery. But let's consider a more interesting case: the conjunctive normal form (CNF) for propositional logic. It is well known that every propositional formula is *logically equivalent* to a propositional formula which is a conjunction of disjunctions of literals. The DIMACS format [6], which is the standard input format for propositional theorem provers nowadays, is just defined in terms of CNF: represent each propositional symbol by a unique positive natural number, use -n to represent $\neg p$ if n represents p, and separate each conjunct with a 0. Put it another way, when using any current propositional theorem prover we can restrict ourselves to a very simple input format containing just integers that represent formulas in CNF. And we can safely make this restriction because each propositional formula whatsoever is equivalent to a CNF formula. Or can we? What do we know, for example, about the relative *sizes* of the original and the CNF translated formula? Actually, while all propositional formulas can be converted into an equivalent formula in CNF, in some cases this conversion can lead to an exponential blow-up of the size of the resulting translation. On the other hand, we can transform formulas to CNF *with only a linear increase in size* if we introduce new propositional variables, and we are only interested in preserving satisfiability [16].

In Section 3 we will investigate normal forms for a wide range of modal logics [4,5]. The starting point is the observation that some modalities can be *extracted* from under the scope of others. Operators like the universal modality [11] or the satisfaction operator [2] constitute typical examples of such modalities. Intuitively, a formula is in extracted form whenever no modality is in the scope of another from which it can be extracted. It turns out that it is not trivial to properly characterize the concept of "extractability" and part of Section 3 will be devoted to this.

We will show how to compute an extracted form for any modal formula. Our method requires an additional transformation to modal conjunctive normal form and, therefore, the resulting formula may be exponentially bigger than the original one. However, if we only require equi-satisfiability, then every formula has an extracted normal form which is only polynomially bigger than the original formula and can be computed in polynomial time. In some logics, the extracted normal form of a formula may have a substantially smaller modal depth. Since the modal depth of a formula is typically used as an estimation of its complexity, these results suggest more accurate complexity measures.

The results in this paper aim to apply to a diversity of modal languages. To achieve this we introduce first, in Section 2, a coinductive definition of model and show that it constitutes a homogeneous framework in which it is possible to cast many different modal languages, ranging from classical modalities to operators from hybrid and memory logics. In addition we will see that, by using coinductive models, results that had to be proved separately for each different language (but whose proofs were known to be mere routine) now can be proved in a general way. We show, for example, that we can have a unique definition of bisimulation for all these languages, and prove a single invariance-under-bisimulation theorem.

2 A Coinductive Take on Modal Operators

Let us start by defining the modal language we will be using throughout the paper.

Definition 2.1 [Modal formula] A modal signature $\mathcal{S}$ is a pair $\mathcal{S} = \langle \mathsf{Atom}, \mathsf{Mod} \rangle$ where Atom and Mod are two countable, disjoint sets. We will usually assume that Atom is infinite. The set of modal formulas over the signature $\mathcal{S} = \langle \mathsf{Atom}, \mathsf{Mod} \rangle$ is defined as

$$\varphi ::= a \mid \neg\varphi \mid \varphi \vee \varphi \mid [m]\varphi,$$

where $a \in \mathsf{Atom}$ and $m \in \mathsf{Mod}$. $\top$ and $\bot$ stand for an arbitrary tautology and contradiction, respectively. We will also use classical connectives such as $\wedge, \rightarrow$ and $\langle m \rangle$, taken to be defined in the usual way.

Let $\mathsf{Sub}(\varphi)$ be the set of subformulas of φ, defined in the usual way. We will write $\varphi(\psi)$ to indicate that $\psi \in \mathsf{Sub}(\varphi)$). We will write $\varphi(\psi/\theta)$ for the formula obtained by uniformly substituting all appearances of ψ in φ by θ.

The language we just defined is exactly the same language of the basic multi-modal logic, but as we will now see, we will be able to cast other modal logics including, for example, hybrid and memory operators right into this same language, in a very natural way. How we do this will become clear once we provide our definition of models.

For convenience, we will use *pointed* models. Typically, a (pointed) Kripke model is defined as a tuple $\langle w, \langle W, R, V \rangle \rangle$ where W is a non-empty set of states, $w \in W$, $R : \mathsf{Mod} \times W \rightarrow 2^W$ defines for each modality m a binary relation over W, and $V : \mathsf{Atom} \rightarrow 2^W$ is a valuation function. In our case, atoms will be interpreted as usual; the difference will be in the way we handle m-successors. For each modality m and each state w in a model, we will define $R(m, w)$, the successors of w through the m modality, as a set of *models* (and not as a set of states in the domain). Therefore, our definition of models will be coinductive.

Definition 2.2 [Models] Let $\mathcal{S} = \langle \mathsf{Atom}, \mathsf{Mod} \rangle$ be a modal signature and W be a fixed, non-empty set. $\mathbf{Mods}_W$, the class of all models with domain W, for the signature $\mathcal{S}$, is the set of all tuples $\langle w, W, V, R \rangle$ such that:

$$\begin{aligned} w &\in W & \\ V(a) &\subseteq W & \text{for } a \in \mathsf{Atom} \\ R(m, w) &\subseteq \mathbf{Mods}_W & \text{for } m \in \mathsf{Mod} \text{ and } w \in W. \end{aligned}$$

$\mathbf{Mods}$ denotes the class of all models over all domains, i.e., $\mathbf{Mods} = \bigcup_W \mathbf{Mods}_W$.

We will keep traditional practice and call w the *point of evaluation*, W the *domain*, V the *valuation*, and R the *accessibility relation*. For $\mathcal{M}$ an arbitrary model we will often write $|\mathcal{M}|$ for its domain, $w^{\mathcal{M}}$ for its point of evaluation, $V^{\mathcal{M}}$ for its valuation and $R^{\mathcal{M}}$ for its accessibility relation. We will sometimes write $succs^{\mathcal{M}}(m)$ for the set $R^{\mathcal{M}}(m, w^{\mathcal{M}})$ of immediate m-successors of $w^{\mathcal{M}}$.

Observe that for each W, $\mathbf{Mods}_W$ is well-defined (coinductively), and so does $\mathbf{Mods}$, the class of all models. Being the class of all possible models, $\mathbf{Mods}$ enjoys some nice closure properties that will be useful when considering subclasses of **Mods**. In particular, we will be interested in investigating modal classes which are *closed under accessibility relations*.

Definition 2.3 [Extension of a model] Given $\mathcal{M} \in \mathbf{Mods}_W$, let $\mathrm{Ext}(\mathcal{M})$, the *extension of* $\mathcal{M}$, be the smallest subset of $\mathbf{Mods}_W$ that contains $\mathcal{M}$ and is such that if $\mathcal{N} \in \mathrm{Ext}(\mathcal{M})$, then $R^{\mathcal{N}}(m, v) \subseteq \mathrm{Ext}(\mathcal{M})$ for all $m \in \mathsf{Mod}$, $v \in W$.

Definition 2.4 [Closed class] A non-empty class of models $\mathcal{C}$ is *closed under accessibility relations* (we will say that $\mathcal{C}$ is a *closed class*, for short) whenever $\mathcal{M} \in \mathcal{C}$ implies $\mathrm{Ext}(\mathcal{M}) \subseteq \mathcal{C}$.

That is, the extension of a model is simply the class of models reachable via the transitive closure of the union of its accessibility relations; and a class of models $\mathcal{C}$ is closed if for every model $\mathcal{M} \in \mathcal{C}$ the extension of $\mathcal{M}$ is also included in $\mathcal{C}$. Clearly, **Mods** is a closed class and, as we will discuss below, it seems natural to restrict ourselves to investigate only closed classes.

Having properly defined what models are, the definition of the satisfiability relation $\models$ is straightforward:

Definition 2.5 [Semantics] For each $\mathcal{M} = \langle w, W, V, R \rangle$ in **Mods** we define:

$$\begin{array}{ll} \mathcal{M} \models a & \text{iff } w \in V(a) \\ \mathcal{M} \models \neg\varphi & \text{iff } \mathcal{M} \not\models \varphi \\ \mathcal{M} \models \varphi \vee \psi & \text{iff } \mathcal{M} \models \varphi \text{ or } \mathcal{M} \models \psi \\ \mathcal{M} \models [m]\varphi & \text{iff } \mathcal{M}' \models \varphi, \text{ for all } \mathcal{M}' \in R(m, w). \end{array}$$

If $\mathcal{C}$ is a closed class, we write $\mathcal{C} \models \varphi$ whenever $\mathcal{M} \models \varphi$ for every $\mathcal{M}$ in $\mathcal{C}$, and we say that $\Gamma_{\mathcal{C}} = \{\varphi \mid \mathcal{C} \models \varphi\}$ is the *logic* defined by $\mathcal{C}$.

Inspecting the definition above, we can see that the semantic clause for $[m]$ is the classical condition defining a box operator [4]. But if we restrict ourselves to the appropriate class of models, we can actually ensure that $[m]$ behaves in very different ways. Let us see an example.

Example 2.6 [The universal modality A] Given a Kripke model $\mathcal{M} = \langle W, R, V \rangle$ the usual semantic clause for the universal modality A would be

$$\mathcal{M}, w \models \mathsf{A}\varphi \text{ iff } \mathcal{M}, w' \models \varphi, \text{ for all } w' \in W.$$

Instead, let $\mathcal{S} = \langle \mathsf{Atom}, \mathsf{Mod} \rangle$ with $\mathsf{A} \in \mathsf{Mod}$, and let $\mathcal{C}_{\mathsf{A}}$ be the largest class of models in this signature such that

$$\text{if } \mathcal{M} \in \mathcal{C}_{\mathsf{A}}, \text{ then } R^{\mathcal{M}}(\mathsf{A}, w) = \{\langle w', |\mathcal{M}|, V^{\mathcal{M}}, R^{\mathcal{M}} \rangle \mid w' \in |\mathcal{M}|\}.$$

That is, the A-successors of w are those models identical to $\mathcal{M}$ except in that their point of evaluation is an arbitrary element of the domain. Clearly, the semantic condition of $[\mathsf{A}]$ in $\mathcal{C}_{\mathsf{A}}$ (as given in Definition 2.5) coincides exactly with the semantic definition of the universal modality A over standard Kripke models.

By taking suitable classes of models, we can naturally capture many different modal operators. Notice, though, that when defining model classes it seems natural to require the classes to be closed. If a class $\mathcal{C}$ is not closed, the evaluation of some modal formulas on a model in $\mathcal{C}$ might require the inspections of models which are outside the class. Moreover, as it follows from Proposition 2.7 below, every closed class induces a *normal* modal logic [4]. In the rest of the paper we will usually implicitly assume that **every class is closed** and that all its models conform to some particular, but arbitrary, signature. We will only mention these conditions explicitly for additional emphasis.

Proposition 2.7 *Let $\mathcal{C}$ be a (closed) class. Then*

(i) $\Gamma_{\mathcal{C}}$ *contains all instances of propositional tautologies.*

(ii) $\Gamma_{\mathcal{C}}$ *is closed under modus-ponens.*

(iii) $\Gamma_{\mathcal{C}}$ *is closed under necessitation, i.e., if* $\varphi \in \Gamma_{\mathcal{C}}$, $[m]\varphi \in \Gamma_{\mathcal{C}}$.

(iv) $\Gamma_{\mathcal{C}}$ *contains every instance of axiom* K_m*, i.e.,* $[m](\varphi \to \psi) \to ([m]\varphi \to [m]\psi)$.

Proof. Closure under tautologies and modus-ponens follows trivially from the semantics of the boolean operators.

Necessitation is straightforward: suppose $\varphi \in \Gamma_{\mathcal{C}}$, then $\mathcal{M} \models \varphi$ for all $\mathcal{M} \in \mathcal{C}$, but since $\mathcal{C}$ is a closed class this implies $\mathcal{M} \models [m]\varphi$ and consequently $[m]\varphi \in \Gamma_{\mathcal{C}}$. (This is the only case were we need to assume that $\mathcal{C}$ is closed.)

Finally, let $[m](\varphi \to \psi) \to ([m]\varphi \to [m]\psi)$ be an instance of axiom K_m and suppose $\mathcal{M} \models [m](\varphi \to \psi)$ and $\mathcal{M} \models [m]\varphi$, for some $\mathcal{M} \in \mathcal{C}$. It follows that for every $\mathcal{M}' \in R(m, w^{\mathcal{M}})$, $\mathcal{M}' \models \varphi \to \psi$ and $\mathcal{M}' \models \varphi$ both hold and, therefore, $\mathcal{M}' \models \psi$ holds too. Hence, $\mathcal{M} \models [m]\psi$. □

By Proposition 2.7 then, the minimal logic (generated by the class of all possible models) $\Gamma_{\mathbf{Mods}}$ is a normal modal logic. As we will show in Proposition 2.10 below, it coincides with the basic multi-modal logic K. In particular, $\Gamma_{\mathbf{Mods}}$ is closed under uniform substitution, that is:

$$\text{If } \varphi \in \Gamma_{\mathbf{Mods}} \text{ then } \varphi(a/\psi) \in \Gamma_{\mathbf{Mods}} \text{ for } a \in \mathsf{Atom}.$$

But for an arbitrary $\mathcal{C}$, $\Gamma_{\mathcal{C}}$ doesn't need to be closed under uniform substitution.

We define next a number of model classes that we will discuss in the rest of the paper. To simplify definitions we introduce the following piece of notation.

Definition 2.8 [Defining conditions] Predicate P is a *defining condition* for $\mathcal{C}$ whenever $\mathcal{C}$ is the *largest* class such that $\mathcal{M} \in \mathcal{C}$ implies that $P(\mathcal{M})$ holds.

We can use this notation to properly define standard relational modalities.

Definition 2.9 [Relational modalities: the classes $\mathcal{C}^{\mathsf{K}}_m$ and $\mathcal{C}^{\mathsf{K}}$] For each $m \in \mathsf{Mod}$, let $\mathcal{C}^{\mathsf{K}}_m$ be the class defined by the following defining condition:

$$P^{\mathsf{K}}_m(\mathcal{M}) \iff \forall v \in |\mathcal{M}|, R^{\mathcal{M}}(m, v) \subseteq \{\langle v', |\mathcal{M}|, V^{\mathcal{M}}, R^{\mathcal{M}}\rangle \mid v' \in |\mathcal{M}|\}$$

Observe that P^{K}_m is true of a model $\mathcal{M}$ if every successor of $w^{\mathcal{M}}$ is identical to $\mathcal{M}$ except perhaps on its point of evaluation. We will call m a *relational modality* when it is interpreted in $\mathcal{C}^{\mathsf{K}}_m$. Define the class of models $\mathcal{C}^{\mathsf{K}}$ over the signature $\mathcal{S} = \langle \mathsf{Atom}, \mathsf{Mod} \rangle$ as follows: $\mathcal{M} \in \mathcal{C}^{\mathsf{K}}$ iff for every modality $m \in \mathsf{Mod}$, $\mathcal{M} \in \mathcal{C}^{\mathsf{K}}_m$. That is, all modalities are interpreted in $\mathcal{C}^{\mathsf{K}}$ as relational modalities.

Proposition 2.10 *The following properties hold:*

(i) $\mathcal{C}^{\mathsf{K}} \subset \mathbf{Mods}$.

(ii) $\Gamma_{\mathcal{C}^{\mathsf{K}}}$ *is the basic multi-modal logic* K.

(iii) $\Gamma_{\mathbf{Mods}}$ *is the basic multi-modal logic* K.

Proof. *(i)* It should be clear that $\mathcal{C}^{\mathsf{K}}$ is a strict subset of **Mods**. Take for example a signature $\langle \{p\}, \{m\} \rangle$ and consider models $\mathcal{M}$ and $\mathcal{N}$ in **Mods** where

$$\mathcal{M} = \langle a, \{a\}, \{(m,a) \mapsto \{\mathcal{N}\}\}, \{p \mapsto \{a\}\} \rangle$$
$$\mathcal{N} = \langle a, \{a\}, \{(m,a) \mapsto \{\mathcal{M}\}\}, \{p \mapsto \emptyset\} \rangle.$$

As $\mathcal{M} \in \mathbf{Mods}$ but $\mathcal{M} \notin \mathcal{C}^{\mathsf{K}}$, we have that $\mathcal{C}^{\mathsf{K}} \subset \mathbf{Mods}$.

(ii) We will use the fact that K is complete with respect to the class of all (classical) pointed Kripke models [4]. Now, to every $\mathcal{M} \in \mathcal{C}^{\mathsf{K}}$ we associate a unique pointed Kripke model $f(\mathcal{M}) = \langle w^{\mathcal{M}}, \langle W^{\mathcal{M}}, R, V^{\mathcal{M}} \rangle \rangle$, where the accessibility relation is defined as $R(m,u) = \{v \mid \langle v, W^{\mathcal{M}}, R^{\mathcal{M}}, V^{\mathcal{M}} \rangle \in R^{\mathcal{M}}(m,u)\}$.

Clearly, f is a bijection between $\mathcal{C}^{\mathsf{K}}$ and the class of all pointed Kripke models, such that for every $\mathcal{M} \in \mathcal{C}^{\mathsf{K}}$, $\mathcal{M} \models \varphi$ iff $f(\mathcal{M}) \models_{\mathsf{K}} \varphi$ (where $\models_{\mathsf{K}}$ is classical modal satisfaction). Therefore, $\Gamma_{\mathcal{C}^{\mathsf{K}}} = \mathsf{K}$.

(iii) Since $\mathcal{C}^{\mathsf{K}} \subset \mathbf{Mods}$ we may conclude $\Gamma_{\mathbf{Mods}} \subseteq \Gamma_{\mathcal{C}^{\mathsf{K}}}$. By Proposition 2.7 we conclude $\mathsf{K} \subseteq \Gamma_{\mathbf{Mods}} \subseteq \mathsf{K}$. □

Proposition 2.10 shows that the basic modal logic K can be recast (in two different ways) using the new semantics. The same can be done for many other modal languages, and we will show some examples in the next section, focusing on hybrid and hybrid-related languages.

2.1 Some Modal Logics and their Associated Classes

Consider the signature $\mathcal{S} = \langle \mathsf{Atom}, \mathsf{Mod} \rangle$ where

$$\mathsf{Atom} = \mathsf{Prop} \cup \mathsf{Nom} \cup \{ⓚ\}$$
$$\mathsf{Mod} = \mathsf{Rel} \cup \{\mathsf{A}\} \cup \{@_i \mid i \in \mathsf{Nom}\} \cup \{\downarrow i \mid i \in \mathsf{Nom}\} \cup \{ⓡ, ⓕ, ⓔ\}$$

where $\mathsf{Prop} = \{p_1, p_2, \ldots\}$, $\mathsf{Nom} = \{n_1, n_2, \ldots\}$ and $\mathsf{Rel} = \{r_1, r_2, \ldots\}$ are mutually disjoint, countable infinite sets. In what follows, we will usually be interested in sub-languages of the language defined over $\mathcal{S}$ by Definition 2.1.

We define the following closed classes via their defining conditions. We start revisiting the modality A that we have already discussed in Example 2.6.

Class	Defining condition
$\mathcal{C}_{\mathsf{A}}$	$\mathcal{P}_{\mathsf{A}}(\mathcal{M}) \iff R^{\mathcal{M}}(\mathsf{A}, w) = \{\langle v, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}, R^{\mathcal{M}} \rangle \mid v \in \lvert\mathcal{M}\rvert\}$

In a similar way, we can capture different operators from hybrid logics [2].

Class	Defining condition
$\mathcal{C}_{@_i}$	$\mathcal{P}_{@_i}(\mathcal{M}) \iff R^{\mathcal{M}}(@_i, w) = \{\langle v, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}, R^{\mathcal{M}}\rangle \mid v \in V(i)\}, i \in \mathsf{Nom}$
$\mathcal{C}_{\downarrow i}$	$\mathcal{P}_{\downarrow i}(\mathcal{M}) \iff R^{\mathcal{M}}(\downarrow i, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[i \mapsto \{w\}], R^{\mathcal{M}}\rangle\}, i \in \mathsf{Nom}$
$\mathcal{C}_{\mathsf{Nom}}$	$\mathcal{P}_{\mathsf{Nom}}(\mathcal{M}) \iff V^{\mathcal{M}}(i)$ is a singleton, $\forall i \in \mathsf{Nom}$

We can even include in this format less well known modalities, like the memory logic operators investigated in [1]:

Class	Defining condition
$\mathcal{C}_{Ⓡ}$	$\mathcal{P}_{Ⓡ}(\mathcal{M}) \iff R^{\mathcal{M}}(Ⓡ, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[Ⓚ \mapsto V^{\mathcal{M}}(Ⓚ) \cup \{w\}], R^{\mathcal{M}}\rangle\}$
$\mathcal{C}_{Ⓕ}$	$\mathcal{P}_{Ⓕ}(\mathcal{M}) \iff R^{\mathcal{M}}(Ⓕ, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[Ⓚ \mapsto V^{\mathcal{M}}(Ⓚ) \setminus \{w\}], R^{\mathcal{M}}\rangle\}$
$\mathcal{C}_{Ⓔ}$	$\mathcal{P}_{Ⓕ}(\mathcal{M}) \iff R^{\mathcal{M}}(Ⓔ, w) = \{\langle w, \lvert\mathcal{M}\rvert, V^{\mathcal{M}}[Ⓚ \mapsto \emptyset], R^{\mathcal{M}}\rangle\}$

Notice, though, that the defining conditions we introduced above are of different kinds. Predicates $\mathcal{P}_{\mathsf{A}}$ and $\mathcal{P}_{@_i}$, for instance, define the accessibility relation by imposing conditions on the point of evaluation of the accessible models (and hence, the classes defined this way are subclasses of the class of relational modalities). This is just another way to state the fact that the semantics of the universal modality A, and satisfiability operators $@_i$ can all be captured on Kripke models by restricting evaluation to the class of models where the relation is, respectively, the total relation ($\forall xy.R(x,y)$) and the 'point to all i' relation ($\forall xy.R(x,y) \leftrightarrow i(y)$). Observe that whenever the atom i is interpreted as a singleton set, the 'point to all i' relation becomes the usual 'point to i' relation ($\forall xy.R(x,y) \leftrightarrow y = i$) of hybrid logics.

Predicate $\mathcal{P}_{\downarrow i}$, on the other hand, imposes conditions on the valuation. In particular, $\mathcal{C}_{\downarrow i}$ is not a subclass of $\mathcal{C}^{\mathsf{K}}_{\downarrow i}$ (that is, the class where $[\downarrow i]$ would be interpreted as a relational modality). For example, uniform substitution fails for $\mathcal{C}_{\downarrow i}$: while it is clear that $\mathcal{C}_{\downarrow i} \models [\downarrow i]i$, the uniform substitution of atom i by p yields the formula $[\downarrow i]p$ which is not $\mathcal{C}_{\downarrow i}$-valid.

Finally, predicate $\mathcal{P}_{\mathsf{Nom}}$ turns elements of Nom into nominals, i.e., true at a unique element of the domain of the model. Again, because $\mathcal{P}_{\mathsf{Nom}}$ imposes conditions the valuation, unrestricted uniform substitution fails.

An interesting feature of this setting is that we can express the combination of modalities as the intersection of their respective classes. For example, $\mathcal{C}_{\mathcal{H}(@,\downarrow)}$, the class of models for the hybrid logic $\mathcal{H}(@,\downarrow)$, can be defined as follows:

$$\mathcal{C}_{\mathcal{H}(@,\downarrow)} = \mathcal{C}_{\mathsf{Nom}} \cap \mathcal{C}_{@} \cap \mathcal{C}_{\downarrow} \cap \mathcal{C}_{\mathsf{Rel}}, \text{ where}$$
$$\mathcal{C}_{@} = \textstyle\bigcap_{i \in \mathsf{Nom}} \mathcal{C}_{@_i},\ \mathcal{C}_{\downarrow} = \bigcap_{i \in \mathsf{Nom}} \mathcal{C}_{\downarrow i}, \text{ and } \mathcal{C}_{\mathsf{Rel}} = \bigcap_{m \in \mathsf{Rel}} \mathcal{C}^{\mathsf{K}}_{m}.$$

Moreover, with this presentation we can define potentially interesting logics which have not been investigated before. For example, consider the logic of $\mathcal{C}_{@}$ (that is, we take $[@_i]$ to be a jump-to-i operator but we do not restrict i to be interpreted as a singleton). Over this class, the $[@_i]$ operator behaves differently than the hybrid operator $@_i$. For example, $\mathcal{C}_{@} \not\models [@_i]\varphi \leftrightarrow \langle @_i\rangle\varphi$; that is $@_i$ is not self dual. But, $\mathcal{C}_{\mathsf{Nom}} \cap \mathcal{C}_{@} \models [@_i]\varphi \leftrightarrow \langle @_i\rangle\varphi$.

Unless otherwise stated, from here on we assume that all operators for which we introduced a special notation (i.e., A, $@_i$, $\downarrow i$, ⓚ, ⓡ, ⓕ, ⓔ and nominals) are interpreted on classes of models that satisfy the corresponding defining conditions introduced in this section.

2.2 Coinductive Models and Bisimulation

We will now turn to the notion of bisimulation, which is central in modal model theory. We will notice that also here the coinductive take on models brings interesting surprises.

Definition 2.11 [Bisimulations] Given two models $\mathcal{M}$ and $\mathcal{M}'$ we say that $\mathcal{M}$ *and* $\mathcal{M}'$ *are bisimilar* (notation $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$) if $\mathcal{M}\, Z\, \mathcal{M}'$ for some relation $Z \subseteq \mathrm{Ext}(\mathcal{M}) \times \mathrm{Ext}(\mathcal{M}')$ such that if $\langle w, W, V, R\rangle\, Z \langle w', W', V', R'\rangle$ then:

(i) $w \in V(a)$ iff $w' \in V'(a)$, for all $a \in \mathsf{Atom}$, [Harmony]

(ii) $\mathcal{N} \in R(m, w)$ implies $\mathcal{N}\, Z\, \mathcal{N}'$ for some $\mathcal{N}' \in R'(m, w')$, [Zig]

(iii) $\mathcal{N}' \in R'(m, w')$ implies $\mathcal{N}\, Z\, \mathcal{N}'$ for some $\mathcal{N} \in R(m, w)$. [Zag]

Such Z is called a *bisimulation between* $\mathcal{M}$ *and* $\mathcal{M}'$.

The classic result of invariance of modal formulas under bisimulation [4] can easily be proved.

Theorem 2.12 *If* $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$*, then* $\mathcal{M} \models \varphi$ *iff* $\mathcal{M}' \models \varphi$*, for all* φ*.*

It is interesting to observe that this very general notion of bisimulation works for every modal logic definable as a closed subclass of **Mods**. In other words, this definition is capturing a variety of notions of bisimulation. Many well known bisimulations can be seen as specializations of Definition 2.11.

Example 2.13 [Bisimulation for $\mathcal{H}(@)$] Consider the hybrid logic $\mathcal{H}(@)$. It is well known that if we want formulas of $\mathcal{H}(@)$ to be preserved by a bisimulation between hybrid models $\mathcal{M}$ and $\mathcal{N}$, we need to require (in addition to Harmony, Zig and Zag) that the bisimulation extends the relation $\{(i^{\mathcal{M}}, i^{\mathcal{N}}) \mid i \in \mathsf{Nom}\}$, where $i^{\mathcal{M}}$ (respectively $i^{\mathcal{N}}$) is the unique state of $\mathcal{M}$ (respectively $\mathcal{N}$) where i is true. Consider now the closed class

$$\mathcal{C}_{\mathcal{H}(@)} = \mathcal{C}_{\mathsf{Nom}} \cap \mathcal{C}_{@} \cap \mathcal{C}_{\mathsf{Rel}}$$

which corresponds (by an argument similar to the one used in the proof of Proposition 2.10) to the class of (pointed) hybrid Kripke models.

Now, suppose we have $\mathcal{M} \underline{\leftrightarrow} \mathcal{N}$ for $\mathcal{M}, \mathcal{N} \in \mathcal{C}_{\mathcal{H}(@)}$. That means $\mathcal{M} Z \mathcal{N}$, for some bisimulation Z (cf. Definition 2.11). We have to show that for all nominals i, $\langle i^{\mathcal{M}}, |\mathcal{M}|, V^{\mathcal{M}}, R^{\mathcal{M}}\rangle\, Z \langle i^{\mathcal{N}}, |\mathcal{N}|, V^{\mathcal{N}}, R^{\mathcal{N}}\rangle$ holds (and is well defined).

The defining condition $\mathcal{C}_{\mathsf{Nom}}$ ensures that the interpretation of i in each model is a singleton, and hence $i^{\mathcal{M}}$ and $i^{\mathcal{N}}$ are well defined. Now, the defining condition $\mathcal{C}_{@}$ together with $\mathcal{M}\, Z\, \mathcal{N}$ let us infer $\langle i^{\mathcal{M}}, |\mathcal{M}|, V^{\mathcal{M}}, R^{\mathcal{M}}\rangle\, Z \langle i^{\mathcal{N}}, |\mathcal{N}|, V^{\mathcal{N}}, R^{\mathcal{N}}\rangle$ using either Zig or Zag.

We can even take a finer look, and adapt the notion of k-bisimulations to the present setting.

Definition 2.14 [k-bisimulations] Given two models $\mathcal{M}$ and $\mathcal{M}'$ we say that *$\mathcal{M}$ and $\mathcal{M}'$ are k-bisimilar* (notation $\mathcal{M} \underline{\leftrightarrow}_k \mathcal{M}'$) if there exists a sequence of binary relations $Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_k$ such that $\mathcal{M}\, Z_k\, \mathcal{M}'$ and for all $\mathcal{N} = \langle w, W, V, R\rangle \in \mathrm{Ext}(\mathcal{M})$ and $\mathcal{N}' = \langle w', W', V', R'\rangle \in \mathrm{Ext}(\mathcal{M}')$:

(i) $\mathcal{N}\, Z_0\, \mathcal{N}'$ implies $w \in V(a)$ iff $w' \in V'(a)$, for all $a \in \mathsf{Atom}$,

(ii) $\mathcal{N}\, Z_{i+1}\, \mathcal{N}'$ and $\mathcal{N}_2 \in R(m, w)$ implies $\mathcal{N}_2\, Z_i\, \mathcal{N}_2'$ for some $\mathcal{N}_2' \in R'(m, w')$,

(iii) $\mathcal{N}\, Z_{i+1}\, \mathcal{N}'$ and $\mathcal{N}_2' \in R'(m, w')$ implies $\mathcal{N}_2\, Z_i\, \mathcal{N}_2'$ for some $\mathcal{N}_2 \in R(m, w)$.

Such a sequence is called a *k-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$*.

One of the interesting properties of k-bisimulations is that they preserve modal formulas up to a certain *modal depth*.

Definition 2.15 [Modal depth] The *modal depth* of a formula φ ($\mathsf{md}(\varphi)$) is a function from formulas to natural numbers defined as:

$$\begin{aligned}
\mathsf{md}(a) &= 0, \text{ for } a \in \mathsf{Atom} \\
\mathsf{md}(\neg\varphi) &= \mathsf{md}(\varphi) \\
\mathsf{md}(\varphi \vee \psi) &= \max\{\mathsf{md}(\varphi), \mathsf{md}(\psi)\} \\
\mathsf{md}([m]\varphi) &= 1 + \mathsf{md}(\varphi).
\end{aligned}$$

The following is a well-known result that transfer nicely to the coinductive setting (once more, it now applies to any type of modality, on any closed class, and not only to relational modalities).

Theorem 2.16 (Invariance under k-bisimulations) *If $\mathcal{M} \underline{\leftrightarrow}_k \mathcal{M}'$, then $\mathcal{M} \models \varphi$ iff $\mathcal{M}' \models \varphi$, for all φ such that $\mathsf{md}(\varphi) \leq k$.*

We have now in place all the tools we need to tackle the main technical results of the paper: normal forms for modal logics. Proposition 2.16 shows that there is a tight connection between modal depth and k-bisimulations. One of the main aims of the next section is to show that the result can be made even tighter.

We will show that each modal formula has an equivalent formula in normal form whose modal depth is always equal or smaller than its own modal depth. For some model classes, the difference in modal depth can be made arbitrarily large. If we are interested in equi-satisfiability and not in equivalency, then we can provide an equi-satisfiable formula with the same property which can moreover be computed in polynomial time (but it will contain additional propositional symbols).

3 Extracted Normal Forms

When working with the basic modal logic, it is standard to take the modal depth of a formula as a measure of its complexity. For example, we know that when searching for a model of a formula φ in the basic modal logic, it is enough to consider models which are trees of depth at most the modal depth of φ. We want to have a closer look at this notion, in the general set up that we presented in the previous section.

Consider, for example, the universal modality A (cf., Example 2.6). If Rel is a set of relational modalities (cf. Definition 2.9), then every formula in the signature $\langle \mathsf{Atom}, \mathsf{Rel} \cup \{\mathsf{A}\}\rangle$ is equivalent to a formula where the $[\mathsf{A}]$ operator only appears at modal depth zero. It suffices to verify the following equivalence:

$$\varphi([\mathsf{A}]\psi) \leftrightarrow \big([\mathsf{A}]\psi \rightarrow \varphi([\mathsf{A}]\psi/\top)\big) \wedge \big(\neg[\mathsf{A}]\psi \rightarrow \varphi([\mathsf{A}]\psi/\bot)\big). \tag{1}$$

In other words, we can check $[\mathsf{A}]\psi$ once and for all at an arbitrary state in the model, and depending of the outcome replace it by $\top$ or $\bot$ throughout the formula. This kind of behavior is not particular to the universal modality; the $@_i$ operators behave in a similar way (independently even of whether i is a nominal or not). And indeed we have that, for Rel a set of relational modalities, every formula in the signature $\langle \mathsf{Atom}, \mathsf{Rel} \cup \{@_i\}\rangle$ is equivalent to a formula where $[@_i]$ only appears at modal depth zero, witness the equivalence:

$$\varphi([@_i]\psi) \leftrightarrow \big([@_i]\psi \rightarrow \varphi([@_i]\psi/\top)\big) \wedge \big(\neg[@_i]\psi \rightarrow \varphi([@_i]\psi/\bot)\big). \tag{2}$$

Let us come back, then, to the idea of using modal depth as a complexity measure. Clearly, if the language contains operators which behaves like $[\mathsf{A}]$ and $[@_i]$ we will have to do better than just count the maximum nesting of operators. If we know that these modalities can be 'extracted' from the scope of other operators, we should rather consider some complexity measure that takes this into account. For example, we know that every formula containing only the $[@_i]$ modality, is equivalent to a formula of modal depth one. Saying that the complexity of $[@_i]^n p$, for example, is n seems way out of line ($[m]^n\varphi$ is the formula φ prefixed by n $[m]$ operators).

But deciding when a modality can be extracted is not trivial. Consider the following examples:

Example 3.1 Let $\mathcal{M}$ be such that $w^{\mathcal{M}} \in V^{\mathcal{M}}(p)$ and $V^{\mathcal{M}}(i) \not\subseteq V^{\mathcal{M}}(p)$. Then we have $\mathcal{M} \models [\downarrow i](p \wedge [@_i]p)$ while $\mathcal{M} \not\models \neg[@_i]p \rightarrow [\downarrow i](p \wedge \bot)$ (since $\mathcal{M} \not\models [@_i]p$) which contradicts (2).

Example 3.2 Let $\mathcal{C} \subseteq \mathcal{C}_{ⓡ} \cap \mathcal{C}_{\mathsf{A}}$. It is easy to verify that $\mathcal{C} \models [ⓡ]\neg[\mathsf{A}]\neg ⓚ$. Now, for every $\mathcal{M} \in \mathcal{C}$ such that $V^{\mathcal{M}}(ⓚ) = \emptyset$, it follows that $\mathcal{M} \not\models [\mathsf{A}]\neg ⓚ \rightarrow [ⓡ]\neg\top$, contradicting (1).

In other words, 'extractability' is not a property of a single modality, but rather of the whole language interpreted in a given class of models. We will provide a definition of extractability in the next section, and investigate ways of determining when a modality can be extracted. Our final goal will be to define a notion of *extracted modal depth* that closer reflects the complexity of a formula.

3.1 Extractable Modalities

We have seen that *extractability* is not a simple notion to characterize, especially when the language contains non-relational modalities like $\downarrow i$ or ⓡ which can block the extraction. But what do we mean when we say that a modality can be extracted from another? Let us start by defining when a modality has *not* been extracted.

Definition 3.3 [Immediate scope] Given two modalities m and n and a formula φ, we say that n *occurs in* φ *in the immediate scope of* m if and only if in the syntactic formation tree of φ there are two nodes e_1 and e_2 such that e_1 is labeled $[m]$, e_2 is labeled $[n]$, e_1 is an ancestor of e_2, and in the path from e_1 to e_2 there are only boolean operators.

Now, if n is $\mathcal{C}$-extractable from m, then every formula φ is equivalent in $\mathcal{C}$ to a formula where n does not occurs in the immediate scope of of m. Formally,

Definition 3.4 We say that n *is* $\mathcal{C}$*-extractable from* m if for any formula φ there exists a formula φ' such that n is not in the immediate scope of m in φ' and $\mathcal{C} \models \varphi \leftrightarrow \varphi'$.

Definition 3.4 does the job when the language contains only two modalities, but consider the following example:

Example 3.5 Take any class of models $\mathcal{C}$ over a signature $\mathcal{S} = \langle \mathsf{Prop}, \{m, n\}\rangle$ such that n is not $\mathcal{C}$-extractable from m (e.g., $\mathcal{C}$ can be the class of relational models over $\mathcal{S}$). Let $\mathcal{S}' = \langle \mathsf{Prop}, \{m, n, n'\}\rangle$ and let $\mathcal{C}'$ be the class of models over $\mathcal{S}'$ such that $\langle W, R', V, n\rangle \in \mathcal{C}'$ iff $\langle W, R, V, n\rangle \in \mathcal{C}$, where $R'(m, x) = R(m, x)$ and $R'(n, x) = R'(n', x) = R(n, x)$, for all $x \in W$. That is, the models of $\mathcal{C}'$ are obtained from those of $\mathcal{C}$ by interpreting n' exactly like n.

Clearly, n is $\mathcal{C}'$-extractable from m because any formula φ containing $[n]$ is equivalent in $\mathcal{C}'$ to the formula φ' obtained by replacing each appearance of $[n]$ by $[n']$ and in φ' no n is in the immediate scope of m. Symmetrically, n' is also $\mathcal{C}'$-extractable from m by replacing all appearances of $[n']$ by $[n]$. But it is not true that every formula is $\mathcal{C}'$-equivalent to one where *neither* n *nor* n' occur in the immediate scope of m, or otherwise n would be $\mathcal{C}$-extractable from m, which would be a contradiction.

We need then a slightly more involved definition of extractability:

Definition 3.6 [$\mathcal{C}$-extractability] Let $S \subseteq \mathsf{Mod} \times \mathsf{Mod}$ be a set of pairs of modalities. We say that S *is* $\mathcal{C}$*-extractable* if for any formula φ there exists a formula φ' such that

(i) for all $(m, n) \in S$ we have that n is not in the immediate scope of m in φ',

(ii) $\mathcal{C} \models \varphi \leftrightarrow \varphi'$.

Whenever S is a singleton set such that $(m, n) \in S$ then S is $\mathcal{C}$-extractable iff n is $\mathcal{C}$-extractable from m.

Notice that the notion of $\mathcal{C}$-extractability is defined in terms of a set of pairs of modalities and for a given model class. Extractability depends not only on the modalities but on the full language (i.e., in which other operators can appear in a formula) and this is captured by restricting the definition to a given class $\mathcal{C}$.

Example 3.7 Let us consider again the signatures and classes of models of Example 3.5. While $\{(m, n)\}$ and $\{(m, n')\}$ are both $\mathcal{C}'$-extractable, $\{(m, n), (m, n')\}$ is not $\mathcal{C}'$-extractable.

Although extractability depends on the full language, the good news is that once you know that a modality is extractable from another, the property is preserved when considering a more expressive logic. This follows directly from Definition 3.6.

If m is $\mathcal{C}$-extractable from n and $\mathcal{C}' \subseteq \mathcal{C}$ then m is $\mathcal{C}'$-extractable from n too. For example, let $\mathcal{C}_1 = \mathcal{C}_{\mathsf{A}} \cap \mathcal{C}^K_m$ and assume we already proved that A is $\mathcal{C}_1$-extractable from m. If we want to add $\downarrow i$ to the language we will be working in the class $\mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C}_{\downarrow i}$, but since $\mathcal{C}_2 \subset \mathcal{C}_1$ we still know A is $\mathcal{C}_2$-extractable from m.

In the rest of this section, we will discuss some sufficient conditions that will ensure $\mathcal{C}$-extractability.

The first sufficient condition we will mention can be motivated by the following observation: if the truth value of an arbitrary $[n]\varphi$ does not change when moving from one state to an m-successor in $\mathcal{C}$, then n should be extractable from m.

Definition 3.8 [m-invariant in $\mathcal{C}$] We say that n *is m-invariant in* $\mathcal{C}$ whenever for every $\mathcal{M} \in \mathcal{C}$, every $\mathcal{N} \in succs^{\mathcal{M}}(m)$ and every formula φ, $\mathcal{M} \models [n]\varphi$ iff $\mathcal{N} \models [n]\varphi$.

We can then prove the following result:

Proposition 3.9 *Let m and n be two modalities and let $\mathcal{C}$ be a class of models. The following two conditions are equivalent:*

(i) n is m-invariant in $\mathcal{C}$

(ii) $\mathcal{C} \models ([n]\varphi \to [m][n]\varphi) \wedge (\langle m\rangle[n] \to [n]\varphi)$, for all φ.

Moreover, if n is m-invariant in $\mathcal{C}$ then n is $\mathcal{C}$-extractable from m.

Proof. We only prove that (i) and (ii) are equivalent. The fact that whenever n is m-invariant in $\mathcal{C}$ then n is $\mathcal{C}$-extractable from m will be a corollary of Theorem 3.20.

$[(i) \Rightarrow (ii)]$ Assume (i), we have to prove $\mathcal{M} \models [n]\varphi \to [m][n]\varphi$ and $\mathcal{M} \models \langle m\rangle[n] \to [n]\varphi$. For the first one, assume $\mathcal{M} \models [n]\varphi$ then (i) directly implies $\mathcal{M} \models [m][n]\varphi$. For the second, assume $\mathcal{M} \models \langle m\rangle[n]\varphi$ and let $\mathcal{N} \in succs^{\mathcal{M}}(m)$ be such that $\mathcal{N} \models [n]\varphi$, again (i) directly implies $\mathcal{M} \models [n]\varphi$

$[(ii) \Rightarrow (i)]$ Assume (ii), and let $\mathcal{N} \in succs^{\mathcal{M}}(m)$. If $\mathcal{M} \models [n]\varphi$ then $\mathcal{M} \models [m][n]\varphi$ and $\mathcal{N} \models [n]\varphi$. If $\mathcal{N} \models [n]\varphi$ then $\mathcal{M} \models \langle m\rangle[n]\varphi$ and hence $\mathcal{M} \models [n]\varphi$. □

Saying that n is m-invariant in $\mathcal{C}$ is strictly stronger than saying that n is $\mathcal{C}$-extractable from m (and it is for that reason that we say that m-invariance is only a necessary condition for extractability). But the additional strength carried out by this notion results in some useful properties. In particular, invariance of a set of pairs of modalities implies its extractability.

Proposition 3.10 *Let $\mathcal{C}$ be a class of models, and let S be a set of pairs of modalities such that $(m, n) \in S$ implies that n is m-invariant in $\mathcal{C}$. Then S is $\mathcal{C}$-extractable.*

Proof. The result follows from Theorem 3.20. □

Notice that Proposition 3.10 is not true if we only require that for each pair $(m, n) \in S$, n is $\mathcal{C}$ extractable from m, as shown in Example 3.7.

Fix a class of models $\mathcal{C}$, and let S be the set of all pairs (m, n) such that n is m-invariant in $\mathcal{C}$. Proposition 3.10 says that any formula φ is $\mathcal{C}$-equivalent to a formula φ' where no n occurs in the immediate scope of m for $(m, n) \in S$. But this implicitly defines a normal form for $\mathcal{C}$, one where all the modalities that are extractable (because of m-invariance) are in fact extracted. Example 3.7 shows that we cannot derive a similar normal form from general extractability.

Definition 3.11 [$\mathcal{C}$-extracted form] We say that φ is in *$\mathcal{C}$-extracted normal form* (or *$\mathcal{C}$-extracted form* for short) if for every n that occurs in φ in the immediate scope of m, n is *not* m-invariant in $\mathcal{C}$.

We will see in Section 3.2 how to compute $\mathcal{C}$-extracted forms. Before that, we will show that it is possible to give (sufficient) conditions on classes of models that ensure m-invariance. That is, we can define conditions under which we may guarantee m-invariance by looking only at the accessibility relation for the involved modalities (i.e., disregarding the rest of the language). Item (*ii*) in Proposition 3.9 gives us a hint for this purely structural characterization.

Definition 3.12 [$\mathcal{C}$-transitive and $\mathcal{C}$-euclidean pair] Given modalities m and n, we say that (m, n) *is a $\mathcal{C}$-transitive pair* whenever for every $\mathcal{M} \in \mathcal{C}$ and every $\mathcal{N} \in \mathit{succs}^{\mathcal{M}}(m)$, $\mathit{succs}^{\mathcal{N}}(n) \subseteq \mathit{succs}^{\mathcal{M}}(n)$; and we say that (m, n) *is a $\mathcal{C}$-euclidean pair* if for every $\mathcal{M} \in \mathcal{C}$ and every $\mathcal{N} \in \mathit{succs}^{\mathcal{M}}(m)$, $\mathit{succs}^{\mathcal{M}}(n) \subseteq \mathit{succs}^{\mathcal{N}}(n)$.

If the pair (m, m) is $\mathcal{C}$-transitive we will just say that m is $\mathcal{C}$-transitive and, similarly, we will say that m is $\mathcal{C}$-euclidean if (m, m) is $\mathcal{C}$-euclidean.

Proposition 3.13 *If (m, n) is a $\mathcal{C}$-transitive and $\mathcal{C}$-euclidean pair, then n is m-invariant in $\mathcal{C}$ (and, hence, $\mathcal{C}$-extractable from m).*

Proof. Suppose $\mathcal{M} \in \mathcal{C}$ and $\mathcal{N} \in \mathit{succs}^{\mathcal{M}}(m)$ and since m and n form a $\mathcal{C}$-pseudo-transitive and a $\mathcal{C}$-pseudo-euclidean pair we know $\mathit{succs}^{\mathcal{N}}(n) = \mathit{succs}^{\mathcal{M}}(n)$, from which it follows that $\mathcal{M} \models [n]\varphi$ iff $\mathcal{N} \models [n]\varphi$. □

There are cases where even simpler conditions can be given.

Definition 3.14 [$\mathcal{C}$-relational and $\mathcal{C}$-constant] We say that a modality m is *$\mathcal{C}$-relational* whenever $\mathcal{C} \subseteq \mathcal{C}^K_m$, and we say that m is *$\mathcal{C}$-constant* if $R^{\mathcal{M}}(m, u) = R^{\mathcal{M}}(m, v)$, for all $\mathcal{M} \in \mathcal{C}$ and all $u, v \in |\mathcal{M}|$.

Example 3.15 As an example of a modality that is constant but not relational, we can consider the reset_i modality (for some $i \in \mathsf{Nom}$) with respect to the class $\mathcal{C}_{\mathsf{reset}_i} \subset \mathcal{C}_{\mathsf{Nom}}$ given by the following defining condition:

$$\begin{aligned} P_{\mathsf{reset}_i}(\mathcal{M}) \quad \iff \quad & R^{\mathcal{M}}(\mathsf{reset}_i, x) = \{\langle w_0, |\mathcal{M}|, V, R^{\mathcal{M}} \rangle\}, \text{ where} \\ & V(x) = \begin{cases} w_0 & \text{if } x \in \mathsf{Nom} \\ V^{\mathcal{M}}(x) & \text{otherwise} \end{cases} \\ & \{w_0\} = V^{\mathcal{M}}(i). \end{aligned}$$

It is easy to see that reset_i is $\mathcal{C}_{\mathsf{reset}_i}$-constant: the reset_i-successor of $\mathcal{M}$ depends exclusively on $V^{\mathcal{M}}$. And since the valuation of this successor may be different from that of $\mathcal{M}$ it is clear that it is not $\mathcal{C}_{\mathsf{reset}_i}$-relational.

Observe that if $\mathcal{C}' \subseteq \mathcal{C}$ and m is $\mathcal{C}$-relational ($\mathcal{C}$-constant) then m is also $\mathcal{C}'$-relational ($\mathcal{C}$-constant). It is straightforward to verify that $@_i$ is $\mathcal{C}_{@_i}$-relational and $\mathcal{C}_{@_i}$-constant (i needs not be a nominal). Similarly, A is $\mathcal{C}_{\mathsf{A}}$-relational and $\mathcal{C}_{\mathsf{A}}$-constant.

One can prove that if m is $\mathcal{C}$-relational and n is $\mathcal{C}$-constant then (m, n) is a $\mathcal{C}$-transitive and $\mathcal{C}$-euclidean pair, and hence, n is $\mathcal{C}$-extractable from m.

Proposition 3.16 *If m is $\mathcal{C}$-relational and n is $\mathcal{C}$-constant, n is m-invariant in $\mathcal{C}$.*

As particular instances of Proposition 3.16 we obtain the following results (where we call a modality m self $\mathcal{C}$-invariant if m is m-invariant in $\mathcal{C}$).

Corollary 3.17

(i) *If $\mathcal{C} \subseteq \mathcal{C}^K_m \cap \mathcal{C}_{@_i}$, then $@_i$ is m-invariant in $\mathcal{C}$.*

(ii) *If $\mathcal{C} \subseteq \mathcal{C}^K_m \cap \mathcal{C}_{\mathsf{A}}$, then A is m-invariant in $\mathcal{C}$.*

(iii) *If $m \in \{\mathsf{A}, @_i, \downarrow i, ⓔ, ⓡ, ⓕ\}$ then m is self $\mathcal{C}_m$-invariant.*

But Proposition 3.16 is strictly weaker than Proposition 3.13. As an example, it is easy to use Proposition 3.13 to show that ⓔ is $\mathcal{C}_{ⓔ} \cap \mathcal{C}_{ⓡ}$-extractable from ⓡ and $\mathcal{C}_{ⓔ} \cap \mathcal{C}_{ⓕ}$-extractable from ⓕ. None of these modalities is relational.

3.2 Computing Formulas in Extracted Form

In this section we will show that for every class $\mathcal{C}$ there exists a computable function $f_{\mathcal{C}}$ such that $f_{\mathcal{C}}(\varphi)$ is in $\mathcal{C}$-extracted form and $\mathcal{C} \models \varphi \leftrightarrow f_{\mathcal{C}}(\varphi)$, for all φ. Throughout this section we will make use of formulas in modal conjunctive normal form ($\text{CNF}_\Box$).

Definition 3.18 [$\text{CNF}_\Box$] We say that a formula is a $\text{CNF}_\Box$-*literal* when it is of the form a, $\neg a$, $[m]\psi$ or $\neg[m]\psi$, for $a \in \mathsf{Atom}$ and ψ a $\text{CNF}_\Box$-*clause*. We say φ is a $\text{CNF}_\Box$-*clause* whenever φ is a $\text{CNF}_\Box$-literal or is of the form $\psi_1 \vee \psi_2$ with ψ_1 and ψ_2 $\text{CNF}_\Box$-clauses. Finally a $\text{CNF}_\Box$-*formula* is a "conjunction" $\neg(\bigvee_i \neg\psi_i)$ of $\text{CNF}_\Box$-clauses ψ_i.

It follows from this definition that a $\text{CNF}_\Box$-formula φ cannot contain two consecutive negations (i.e., $\neg\neg\psi$ is not a subformula of φ). Moreover, if $[m]\psi$ occurs in a $\text{CNF}_\Box$-formula, then ψ must be either a (negated) atom, a disjunction where no disjunct is a conjunction, or a (negated) formula of the form $[n]\chi$ (with m and n not necessarily different). We use this observation to show that it is simple to obtain a $\mathcal{C}$-extracted formula from one in $\text{CNF}_\Box$.

Lemma 3.19 *For every class $\mathcal{C}$, there exists a computable function $f_{\mathcal{C}}$ such that for all φ in $\text{CNF}_\Box$, $f_{\mathcal{C}}(\varphi)$ is a $\text{CNF}_\Box$-formula in $\mathcal{C}$-extracted form and $\mathcal{C} \models \varphi \leftrightarrow f_{\mathcal{C}}(\varphi)$.*

Proof. Given a class $\mathcal{C}$, let $E_{\mathcal{C}}$ be the rewrite system that contains, for every n that is m-invariant in $\mathcal{C}$, the following rules (modulo commutativity of $\vee$):

$$[m][n]\varphi \longrightarrow_{E_{\mathcal{C}}} [m]\bot \vee [n]\varphi \tag{3}$$
$$[m]\neg[n]\varphi \longrightarrow_{E_{\mathcal{C}}} [m]\bot \vee \neg[n]\varphi \tag{4}$$
$$[m](\psi \vee [n]\varphi) \longrightarrow_{E_{\mathcal{C}}} [m]\bot \vee ([m]\psi \vee [n]\varphi) \tag{5}$$
$$[m](\psi \vee \neg[n]\varphi) \longrightarrow_{E_{\mathcal{C}}} [m]\bot \vee ([m]\psi \vee \neg[n]\varphi) \tag{6}$$

It is not hard to see that $E_{\mathcal{C}}$ is terminating (but not confluent, although this is not relevant here). From the observations above, it is also straightforward to see that if a $\mathrm{CNF}_{\Box}$-formula φ is not in $\mathcal{C}$-extracted form, then there is an occurrence of m and n that satisfies the left-hand-side of one of the rules. Moreover, since the rules preserve clausal form, we can conclude that every $\mathrm{CNF}_{\Box}$-formula is rewritten to a $\mathrm{CNF}_{\Box}$-formula in $\mathcal{C}$-extracted form.

It only remains to see that the resulting formula is $\mathcal{C}$-equivalent; for this, it suffices to show that both sides of each rule are actually $\mathcal{C}$-equivalent. We will discuss $\mathcal{C} \models [m](\psi \vee \neg[n]\varphi) \leftrightarrow [m]\bot \vee ([m]\psi \vee \neg[n]\varphi)$, the other cases are analogous. Suppose, then, that for $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models [m](\psi \vee \neg[n]\varphi)$. It could be the case that $\mathcal{M}$ has no m-successors, or that every m-successor satisfies ψ, but then, clearly, $\mathcal{M} \models [m]\bot \vee [m]\psi$. Suppose, then that for some $\mathcal{N} \in \mathit{succs}^{\mathcal{M}}(m)$, $\mathcal{N} \models \neg\psi \wedge \neg[n]\varphi$. Since n is m-invariant in $\mathcal{C}$, we know $\mathcal{M} \models [n]\varphi$ iff $\mathcal{N} \models [n]\varphi$, so we can conclude that $\mathcal{M} \models \neg[n]\varphi$. The other direction is proved in a similar way. □

We can now prove the main result of this section. Observe that both Proposition 3.10 and the bit of Proposition 3.9 that was left unproven will follow as a trivial corollary.

Theorem 3.20 *For every class $\mathcal{C}$, there exists a translation $f_{\mathcal{C}}$ such that for all φ, $f_{\mathcal{C}}(\varphi)$ is in $\mathcal{C}$-extracted form and $\mathcal{C} \models \varphi \leftrightarrow f_{\mathcal{C}}(\varphi)$.*

Proof. From Lemma 3.19, this proof simply amounts to observing that every formula can be recursively turned to an equivalent one in $\mathrm{CNF}_{\Box}$, which is a folklore result. For the sake of completeness, we include such a computable transformation. Let $\longrightarrow_{\mathrm{CNF}_{\Box}}$ be the rewrite system that contains the following rules (modulo commutativity):

$$\neg\neg\varphi \longrightarrow_{\mathrm{CNF}_{\Box}} \varphi \tag{7}$$
$$\varphi \vee \neg(\psi \vee \chi) \longrightarrow_{\mathrm{CNF}_{\Box}} \neg(\neg(\varphi \vee \neg\psi) \vee \neg(\varphi \vee \neg\chi)) \tag{8}$$
$$[m]\neg(\varphi \vee \psi) \longrightarrow_{\mathrm{CNF}_{\Box}} \neg(\neg[m]\neg\varphi \vee \neg[m]\neg\psi) \tag{9}$$

The last rule, of course, must be instantiated for every $m \in \mathsf{Mod}$. It is easy to see that the left and right-hand-sides of each rule are logically equivalent. Rule (7) eliminates double negations and rules (8) and (9) are simply distribution of disjunction and box, respectively, over conjunctions. If a formula is not in $\mathrm{CNF}_{\Box}$, then it may be rewritten using one of these rules; and since the system can be shown to be terminating, it constitutes the desired computable transformation. □

Of course, since the proof above uses a translation to $\mathrm{CNF}_{\Box}$, the formula in extracted form may end up being exponentially larger than the original one. This begs the question if there exists an alternative translation with only a polynomial overhead. The answer is negative for the general case, as is witnessed by a result from ten Cate [20].

Theorem 3.21 (ten Cate, 2005) *There is no polynomial translation from $\mathcal{H}(@)$-formulas to $\mathcal{C}_{\mathcal{H}(@)}$-equivalent formulas in $\mathcal{C}_{\mathcal{H}(@)}$-extracted form.*

On the other hand, if we restrict ourselves to satisfiability preserving translations, then it is indeed possible to give polynomial translations for arbitrary classes.

Theorem 3.22 *For every class $\mathcal{C}$, there exists a polynomial translation $f_{\mathcal{C}}$ such that for all φ, $f_{\mathcal{C}}(\varphi)$ is in $\mathcal{C}$-extracted form, and φ is satisfiable iff $f_{\mathcal{C}}(\varphi)$ is satisfiable.*

Proof. It is easy to verify that the transformation used in the proof of Lemma 3.19 runs in polynomial time. Hence, we only need to provide a polynomial, satisfiability preserving translation to $\mathrm{CNF}_{\Box}$. Mints exhibits a sequent-based polynomial translation for the case of the basic unimodal logic [14]. We will use instead a rewriting based translation similar to the ones above. Observe that in the proof of Theorem 3.20, the source for the exponential blow-up is in rule (8) where φ occurs twice on its right-hand-side. The rewrite system $\longrightarrow_{\text{poly-CNF}_{\Box}}$ is obtained by replacing this rule by one that uses additional proposition symbols:

$$\neg\neg\varphi \longrightarrow_{\text{poly-CNF}_{\Box}} \varphi \tag{10}$$
$$\varphi \vee \neg(\psi \vee \chi) \longrightarrow_{\text{poly-CNF}_{\Box}} \neg(\neg(\varphi \vee \neg p_{\varphi}) \vee \neg(p_{\varphi} \vee \psi) \vee \neg(\neg p_{\varphi} \vee \chi)) \tag{11}$$
$$[m]\neg(\varphi \vee \psi) \longrightarrow_{\text{poly-CNF}_{\Box}} \neg(\neg[m]\neg\varphi \vee \neg[m]\neg\psi) \tag{12}$$

It is routine to see that $\longrightarrow_{\text{poly-CNF}_{\Box}}$ is terminating (but not confluent, in fact different reduction strategies may introduce different proposition symbols). In any case, it is not hard to see that the length of every derivation is bound by a polynomial. Finally, one needs to show that every rewrite step leads to an equi-satisfiable formula, but this is also straightforward. $\Box$

3.3 Modal depth, invariance under k-bisimulation and formula complexity

The modal depth of a formula is often taken as a measure of its complexity. Its appeal lies in that it estimates and summarizes in a single value several aspects of complexity: the expressive power of the formula (cf. classic invariance under k-bisimulations results of Section 2.2), the computational cost of evaluating the formula in a model, the minimum size of a model for the formula, etc.

Now, suppose $\mathcal{C} \models \varphi \leftrightarrow \varphi'$, and $\mathsf{md}(\varphi) > \mathsf{md}(\varphi')$. It would often make sense to prefer $\mathsf{md}(\varphi')$ as a more accurate estimation of the complexity of φ, specially if the cost of computing $\mathsf{md}(\varphi')$ can be disregarded. Theorem 3.20 tells us that $\mathcal{C} \models \varphi \leftrightarrow f_{\mathcal{C}}(\varphi)$ and $\mathsf{md}(\varphi) \geq \mathsf{md}(f_{\mathcal{C}}(\varphi))$. But, while $\mathsf{md}(f_{\mathcal{C}}(\varphi))$ appears to be a more accurate measure of the complexity of φ than $\mathsf{md}(\varphi)$, there are two drawbacks. First, $f_{\mathcal{C}}$ is not univoquely defined in Theorem 3.20 (since different rewrite strategies lead to different normal forms). Second, even if we fix a definition for $f_{\mathcal{C}}$ (e.g., by fixing a rewrite strategy) we already saw that $f_{\mathcal{C}}(\varphi)$ can be exponentially larger than φ and hence, first computing $f_{\mathcal{C}}(\varphi)$ and then obtaining its modal depth would be too expensive.

In this section, by using a generalization of the notion of modal depth (cf. the definition of "modal paths" below) we will be able to give more accurate invariance results. These results will cover a wide spectrum of complexity measures computable in polynomial time.

As a generalization of modal depth we will take all the sequences of modalities occurring in some branch of the formation tree of a formula. We will call this set the *modal paths* of a formula.

Definition 3.23 [Modal paths] We define $\pi(\varphi) \subset \mathsf{Mod}^*$, the set of *modal paths* of a formula φ, in an inductive way:

$$\begin{aligned}\pi(a) &= \{\epsilon\}\\ \pi(\neg\psi) &= \pi(\psi)\\ \pi(\psi \vee \chi) &= \pi(\psi) \cup \pi(\chi)\\ \pi([m]\psi) &= \{m.p \mid p \in \pi(\psi)\}\end{aligned}$$

Observe that $\mathsf{md}(\varphi) = \max\{|t| \mid t \in \pi(\varphi)\}$, where $|t|$ is the length of t and that $\pi(\varphi)$ can be computed in polynomial time. We will use π to define complexity measures which are more accurate than md.

Definition 3.24 [Extracted variants and k-bounds] Given two finite sets $M, M' \subset \mathsf{Mod}^*$, we say that *$M'$ is a $\mathcal{C}$-extracted variant of M*, notated $M' \in \tau_{\mathcal{C}}(M)$, if M' can be obtained from M by repeatedly applying the following rule until it can no longer be applied (here $s, t \in \mathsf{Mod}^*, m, n \in \mathsf{Mod}$):

$$\frac{A \cup \{s.m.n.t\}}{A \cup \{s.m, s.n.t\}}\ n \text{ is } m\text{-invariant in } \mathcal{C} \tag{13}$$

Finally, we say a formula φ is *k-bounded in $\mathcal{C}$* if there exists $M \in \tau_{\mathcal{C}}(\pi(\varphi))$ such that $k \geq \max\{|t| \mid t \in M\}$.

Any function c will be a correct measure of complexity for $\mathcal{C}$ as long as φ is $c(\varphi)$-bounded in $\mathcal{C}$ for all φ. Trivially, then, md will be correct for all $\mathcal{C}$. The following theorem links k-bounds with k-bisimulations, justifying this claim.

Theorem 3.25 *Let $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ be such that $\mathcal{M} \underline{\leftrightarrow}_k \mathcal{N}$. Then, for all φ k-bounded in $\mathcal{C}$, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.*

Proof. First, observe that, for all ψ in $\mathrm{CNF}_\Box$, if M is obtained from $\pi(\psi)$ by applying rule (13) once, then there exist a derivation $\psi \longrightarrow_{E_\mathcal{C}} \psi_1 \longrightarrow_{E_\mathcal{C}} \cdots \longrightarrow_{E_\mathcal{C}} \psi_n$ such that $\pi(\psi_n) = M'$ (notice that if $n > 1$ then there exist more than one path in the syntactic tree of ψ where the rewritten modal path occurs). Conversely, if $M = \pi(\psi)$ and rule (13) cannot be applied to M, then ψ must be in extracted form.

Now, since φ is k-bounded, there must be a derivation of M from $\pi(\varphi)$ such that $k \geq \max\{|t| \mid t \in M\}$. Let φ' be a $\mathrm{CNF}_\Box$-formula $\mathcal{C}$-equivalent to φ obtained using the rewrite system $\longrightarrow_{\mathrm{CNF}_\Box}$ (cf. Theorem 3.20). It is trivial to see that $\pi(\varphi) = \pi(\varphi')$, but this means that we can use the observation above to derive a $\mathcal{C}$-equivalent φ'' from φ' using $\longrightarrow_{E_\mathcal{C}}$ such that $\pi(\varphi'') = M$ and φ'' is in $\mathcal{C}$-extracted form. Therefore we have $\mathsf{md}(\varphi'') \leq k$ and using Theorem 2.16, $\mathcal{M} \models \varphi''$ iff $\mathcal{N} \models \varphi''$. Finally, since φ, φ' and φ'' are all $\mathcal{C}$-equivalent, we conclude $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$. $\Box$

To illustrate this result, we propose next a (polynomially computable) complexity measure for $\mathcal{H}(@)$ and show it is correct and more accurate than md.

Definition 3.26 [Hybrid depth] The *hybrid depth* of a $\mathcal{H}(@)$-formula φ ($\mathsf{hd}(\varphi)$) is defined as $\mathsf{hd}(\varphi) = \max\{\mathsf{hd}'(\varphi), 1 + \max\{\mathsf{hd}'(\psi) \mid [@_i]\psi \in \mathsf{Sub}(\varphi)\}\}$, where:

$$
\begin{aligned}
\mathsf{hd}'(a) &= 0, \text{ for } a \in \mathsf{Atom} \\
\mathsf{hd}'(\neg\varphi) &= \mathsf{hd}'(\varphi) \\
\mathsf{hd}'(\varphi \vee \psi) &= \max\{\mathsf{hd}'(\varphi), \mathsf{hd}'(\psi)\} \\
\mathsf{hd}'([@_i]\varphi) &= 0 \\
\mathsf{hd}'([r]\varphi) &= 1 + \mathsf{hd}'(\varphi), \text{ for } r \in \mathsf{Rel}.
\end{aligned}
$$

Clearly, hd is computable in polynomial time and $\mathsf{hd}(\varphi) \leq \mathsf{md}(\varphi)$ for all φ. Moreover, for every k there exists φ_k such that $\mathsf{md}(\varphi_k) = \mathsf{hd}(\varphi_k) + k$. In other words, when considering the set of all hybrid formulas, hd can be found to be arbitrarily smaller than md. Still, hd is a correct measure of complexity for $\mathcal{H}(@)$.

Proposition 3.27 *For every $\mathcal{H}(@)$-formula φ, φ is $\mathsf{hd}(\varphi)$-bound in $\mathcal{C}_{\mathcal{H}(@)}$.*

Proof. Observe that rule (13) is confluent in the case of $\mathcal{C}_{\mathcal{H}(@)}$ (it need not be in the general case) so there is only one $M \in \tau_{\mathcal{C}}(\pi(\varphi))$. Moreover, if $s \in M$, then either $s \in \mathsf{Rel}^*$ or $s = @_i.t$ with $t \in \mathsf{Rel}^*$. In the last case it must be because $@_i\psi$ with $t \in \pi(\psi)$ occurs in φ. In any case, it is clear that $\mathsf{hd}(\varphi) = \max\{|s| \mid s \in M\}$.□

As a final application of Theorem 3.25 we exhibit an alternative proof for a classic result in computational complexity for modal logics [13].

Theorem 3.28 (Ladner, 1977) *The satisfiability problem for **S5** is NP-complete.*

Proof. The lower bound is given by the complexity of the satisfiability problem for propositional logic. For the upper bound, assume a signature $\mathcal{S} = \langle \mathsf{Atom}, \{m\}\rangle$ and take $\mathcal{C}_{\mathbf{S5}}$ to be the class of all $\mathcal{S}$-models where m is interpreted as a relational, transitive, reflexive and symmetric modality. Since this class satisfies axioms 5, we know m has to be euclidean too. Suppose we are looking a model for φ. By Proposition 3.13, m is self $\mathcal{C}$-extractable. It is easy to see that then if $M \in \tau_{\mathcal{C}}(\pi(\varphi))$ then M is the singleton set $\{m\}$. Therefore, φ is 1-bounded in $\mathcal{C}_{S5}$ and, therefore it suffices to non-deterministically guess a proper model $\mathcal{M}$ of depth 1 with enough successors (e.g., the number of occurrences of $[m]$ in φ) and verify if $\mathcal{M} \models \varphi$, which can be done in polynomial time. □

4 Conclusions

Conjunctive and clausal normal forms for modal logics can be found in the literature, for example, in the context of automated reasoning. For instance in [7] they are used to define resolution calculi for modal logics. Decision procedures based on propositional SAT (e.g., [9,8]) also take advantage of these forms. In particular, provers like KSat [10] and *SAT [19] transform their input into some kind of modal conjunctive normal form. This normal form also lies behind several methodologies for evaluating performance of modal provers [17]. Finally, the characterization of the *modal Horn fragment* for which several good complexity results are known (e.g. [15]) is also based on modal normal forms.

The original goal of this paper was to investigate normal forms for hybrid logics with an interest in automated deduction.

There is a folklore correlation between the modal depth of a formula and how hard it is for a prover to establish its satisfiability. I.e., on average, the larger the nesting of modalities, the more difficult to find a model or to prove that there is none. But some empirical results obtained while testing satisfiability for *hybrid* formulas in modal conjunctive normal form, pointed out that the standard measure of modal depth (which simply counted the maximum nesting of modal operators) did not seem suitable for formulas containing the @ operator. Formulas containing @ can have a high modal depth, while being satisfied in a small model. A trivial example is the formula $(@_i)^n p$ which in spite of having modal depth n can be easily seen to be satisfied in a single point model. This empirical evidence pointed out the need of defining a more accurate measure of complexity.

That global modal operators like the universal modality and satisfiability operators could be extracted was a well known fact, that is already used in the seminal article of Goranko and Passy [11]. In a different context, normal forms for certain kind of extractable modalities in the presence of the universal modality are also discussed by van de Hoek and de Rijke in [21] (see Definitions 2.6 and 2.9 and related results in that paper). The results presented in this paper are more general and are not restricted to normal forms that are equivalent to the original formula. Concretely, Theorems 3.20 and 3.22 show how to obtain modal conjunctive normal forms which may have a smaller modal depth than the standard definition and which preserve formula equivalence and formula satisfiability, respectively. While the normal form preserving equivalence might result in an exponential blow up (as is already the case with propositional logic), the normal form preserving satisfiability is only polynomially larger. Given this observation, Theorems 3.20 and 3.22 suggest that it may be worth to use the extracted normal form as input instead of φ when trying to decide if φ is $\mathcal{C}$-satisfiable. This is especially significant if the input has to be converted to modal conjunctive normal form in the first place.

In hybrid logics, modalities like satisfiability and universal operators can be *extracted* from under the scope of other operators. We introduce in Definition 3.6 a general notion of extractability, and point to sufficient conditions for extractability in Definition 3.8 by means of the notion of *invariance*. We then show how to compute an extracted conjunctive normal form such that whenever a modality n is invariant for a modality m, n has been extracted from m and does not appear in its immediate scope.

To present the results outlined above in a proper framework, we introduce a coinductive definition of models for modal logics. We show that this new definition provides a homogeneous framework in which it is possible to include many different modal languages ranging from classical modalities to operators from hybrid and memory logics. Moreover, we introduce general notions of bisimulation and k-bisimulation that uniformly apply to all these languages and prove that they preserve modal formulas.

As possible future lines of research we are interested, on the practical side, in further empirical testing of the behaviour of hybrid formulas transformed into extracted normal form. Moreover, the current paper made no attempt to integrate any kind of optimizations in the translation into extracted normal form. It would be possible to define special rules that can further reduce the modal depth of the obtained formula. On the theoretical side, we plan to further investigate the coinductive framework. In particular, it would be interesting to compare it with the extensive work on co-algebraic modal logics (e.g., [12] among others).

References

[1] Areces, C., D. Figueira, S. Figueira and S. Mera, *Expressive power and decidability for memory logics*, in: *Logic, Language, Information and Computation*, Lecture Notes in Computer Science **5110** (2008), pp. 56–68, proceedings of WoLLIC 2008.

[2] Areces, C. and B. ten Cate, *Hybrid logics*, in: Blackburn et al. [5] pp. 821–868.

[3] Baaz, M., U. Egly and A. Leitsch, *Normal form transformations*, in: Robinson and Voronkov [18] pp. 273–333.

[4] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge University Press, 2001.

[5] Blackburn, P., F. Wolter and J. van Benthem, editors, "Handbook of Modal Logics," Elsevier, 2006.

[6] *DIMACS satisfiability suggested format*, `ftp://dimacs.rutgers.edu/pub/challenge/satisfiability/doc` (1993), last visited 22/08/2009.

[7] Enjalbert, P. and L. Fariñas del Cerro, *Modal resolution in clausal form*, Theoretical Computer Science **65** (1989), pp. 1–33.

[8] Giunchiglia, E., A. Tacchella and F. Giunchiglia, *Sat-based decision procedures for classical modal logics*, Journal of Automated Reasoning **28** (2002), pp. 143–171.

[9] Giunchiglia, F. and R. Sebastiani, *Building decision procedures for modal logics from propositional decision procedures: Case study of modal K*, in: *Proceedints of CADE-13*, number 1104 in LNAI, 1996, pp. 583–597.

[10] Giunchiglia, F. and R. Sebastiani, *A SAT-based decision procedure for $\mathcal{ALC}$*, in: *Principals of Knowledge Representation and Reasoning: Proceedings of the Fifth International Conference (KR'96)* (1996), pp. 304–314.

[11] Goranko, V. and S. Passy, *Using the universal modality: Gains and questions*, Journal of Logic and Computation **2** (1992), pp. 5–30.

[12] Kurz, A., *Coalgebras and their logics*, SIGACT News **37** (2006), pp. 57–77.

[13] Ladner, R., *The computational complexity of provability in systems of modal propositional logic*, SIAM Journal on Computing **6** (1977), pp. 467–480.

[14] Mints, G., *Resolution calculi for modal logics*, in: *Eight Papers Translated from the Russian*, American Mathematical Society Translations **143**, AMS, 1989 pp. 1–14.

[15] Nguyen, L., *On the complexity of fragments of modal logics*, in: R. Schmidt, I. Pratt-Hartmann, M. Reynolds and H. Wansing, editors, *Advances in Modal Logic 5* (2004), pp. 249–268.

[16] Nonnengart, A. and C. Weidenbach, *Computing small clause normal forms*, in: Robinson and Voronkov [18] pp. 335–366.

[17] Patel-Schneider, P. and R. Sebastiani, *A new general method to generate random modal formulae for testing decision procedures*, Journal of Artificial Intelligence Research **18** (2003), pp. 351–389.

[18] Robinson, J. and A. Voronkov, editors, "Handbook of Automated Reasoning (in 2 volumes)," Elsevier and MIT Press, 2001.

[19] Tacchella, A., **SAT system description*, in: P. Lambrix, A. Borgida, M. Lenzerini, R. Möller and P. Patel-Schneider, editors, *Proceedings of the 1999 International Workshop on Description Logics (DL'99), Linköping, Sweden, July 30 - August 1, 1999*, CEUR Workshop Proceedings **22** (1999).

[20] ten Cate, B., "Model theory for extended modal languages," Ph.D. thesis, University of Amsterdam (2005), ILLC Dissertation Series DS-2005-01.

[21] van der Hoek, W. and M. de Rijke, *Generalized quantifiers and modal logic*, Journal of Logic, Language and Information **2** (1993), pp. 19–58.