Repairing the Interpolation Theorem in First-Order Modal Logic*

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Introduction. Hybrid logics are extensions of orthodox modal logics in which it is possible to name worlds (or states, or times, ...). Such extensions have proved interesting for a number of reasons. We often want to reason about what happens at particular states, times, or locations, and this isn't possible in orthodox modal logics. In addition, hybridization can improve the behavior of the underlying modal logic. For example, many modally undefinable properties (such as irreflexivity) are definable in hybrid logic, and general completeness and interpolation results can be obtained.

Previous work has examined the effects of hybridizing *propositional* modal logics. But what happens when a *first-order* modal logic is hybridized? First-order modal logics are often badly behaved — does hybridization improve the situation? As far as interpolation is concerned, yes. As we shall see, hybridization allows us to repair failures of interpolation in first-order modal logic, in a very general way.

This abstract is not self contained. For first-order modal logic (and in particular, a discussion of varying, expanding, contracting, and constant domains), see Fitting and Mendelsohn [4]. For Beth definability and Craig iterpolation, see Chang and Keisler [2]. As for hybrid logic, the most relevant points are the following. Hybrid logic contains special variables, written w, w', ..., which range over worlds. Prefixing a formula by $\downarrow w$ binds the variable w to the world of evaluation, while prefixing a formula by $@_w$ means that the formula should be evaluated at the state named by w. As the examples below show, it is the interplay between $\downarrow w$ ("let w name the current state") and $@_w$ ("evaluate at the state named by w") that enable us to create interpolants so easily. For a detailed examination of $\downarrow$ and $@$ in the propositional case, see Areces, Blackburn and Marx [1].

Repairing Fine's Counterexamples. Kit Fine [3] showed that both Beth's definability theorem and Craig's interpolation theorem fail for first-order **S5** without any assumption on the domains, and that both results fail for any first-order modal logic between **K** and **S5** when the constant domain axiom schema is added.

For first-order **S5** with varying domains, Fine provides the following counterexample. Let ϕ be the formula $p \wedge \Box\forall x\Box(p \rightarrow \mathsf{E}x)$, and let ψ be $q \rightarrow \Box\forall x\Diamond(q \wedge \mathsf{E}x)$, where $\mathsf{E}x$ is used as an abbreviation for $\exists y(y = x)$ and may be read as "x exists". Let CON_s mean that for any s' such that Rss', $D_{s'} \subseteq D_s$. Thus CON_s expresses that the domains are contracting from s. Under the assumption of a symmetric accessibility relation R, $\mathfrak{M}, s \models \phi$ implies CON_s and CON_s implies $\mathfrak{M}, s \models \psi$. Thus $\phi \rightarrow \psi$ is a theorem in first-order **S5** and the required interpolant should express CON_s in the empty signature. However, Fine shows that such an interpolant does not exist in first-order **S5**.

But now consider the following hybrid formula: $\downarrow w\Box\forall x @_w \mathsf{E}x$. Recall that $\downarrow w$ binds w to the current world, and that $@_w$ shifts the point of evaluation to the world named by w. So the formula says: at any world accessible from w, any individual x in such a world already exists at world w. Thus it is an interpolant.

*The full version of this abstract will shortly be submitted for journal publication. It will also be made available on the hybrid logic homepage, `http://www.hylo.net`.

For first-order **S5** with constant domains, Fine provides this counterexample to the Beth Definability Theorem: let T be the theory consisting of the following two formulas

$$\begin{array}{lcl} p & \to & \Diamond\forall x(Fx \to \Box(p \to \neg Fx)) \\ \neg p & \to & \Box\exists x(Fx \wedge \Box(\neg p \to Fx)). \end{array}$$

Let $\mathfrak{M}$ be a model, and s any state in $\mathfrak{M}$. Let $\bar{F}_s$ be $\{d \in \mathsf{D} \mid \mathfrak{M}, s \models F(d)\}$. Fine shows that $\mathfrak{M}, s \models p$ iff $\bar{F}_s$ is disjoint from $\bar{F}_{s'}$ for some s' accessible from s. So we have here an implicit definition of p, but Fine shows that there no explicit definition in first-order **S5**, and thus Beth definability (and Craig interpolation) fails.

But in first-order hybrid logic, the required concept is easy to express: in any $\mathfrak{M}$, $\bar{F}_s$ is disjoint from $\bar{F}_{s'}$ for some s' accessible from s iff

$$\mathfrak{M}, s \models {\downarrow} w \Diamond\forall x(Fx \to @_w \neg Fx).$$

This formula names the world of evaluation with $\downarrow w$, and then says: there is some world accessible from w such that for every individual x that exists there, if x has property F in that world, then x does not have property F at world w. This is exactly the required definition.

Interpolation in First-Order Hybrid Logic We shall now state our result. First some terminology. A *frame* is an ordered tuple (W, R), with W a non-empty set of states and R a binary relation on R. A *skeleton* is an ordered triple (W, R, D), with (W, R) a frame and D a function with domain W assigning to each state $s \in W$, a non-empty set D_s. The *bounded fragment* of first-order logic is the set of formulas in one free variable generated by the following grammar:

$$\varphi = Rxy \mid x = y \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \exists x(Ryx \wedge \varphi) \mid \forall x(Ryx \to \varphi) \text{ (for } x \neq y).$$

A class of frames (skeletons) F is *bounded fragment definable* if F consists of all frames (skeletons) satisfying $\forall x Cx$, for Cx some formula in the bounded fragment. Many well known classes of frames (skeletons) are bounded fragment definable. For example, reflexivity, symmetry, transitivity, confluence, and non-branchingness are all bounded fragment definable.

Theorem 1 (Interpolation Theorem) *Let* F *be a bounded fragment definable class of skeletons, with either varying, expanding, contracting or constant domains. Then the first-order hybrid logic of* F *enjoys interpolation.*

The proof (given in the full version of the paper) is model theoretic, much in the style of Chang and Keisler's proof of Craig's theorem. A corollary of the theorem is that all counterexamples mentioned in [3] are repaired. In particular, first-order hybrid **K, T, S4, S5**, all enjoy interpolation, no matter what domain condition we impose.

References

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