Update, the infinite case

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Abstract

We provide a set of postulates that characterizes Katsuno and Mendelzon's update operation [3] for a *possibly infinite* propositional language. In this way we extend their original operator which was only defined for finite languages. After reformulating the update operation as a function on theories, we show that Katsuno and Mendelzon's original postulates are not sufficient to provide a characterization theorem. A strengthening of postulate (U8) is needed for a representation result.

The new definition of update on (possibly infinite) languages puts the two main notions of theory change (update and revision) in the same ground and opens the way to a better comparison of their properties.

Keywords: logic of theory change, update, AGM revision.

1 Preliminaries

The field of Theory Change (or Belief Revision as it is also commonly known) is concerned with the study of ways of modifying a given "state of affairs" or theory, when new information about the situation is available.

In 1985, the seminal paper of Alchourrón, Gärdenfors and Makinson (AGM) [1] provided a specific way of modifying a logical theory given a single formula to be considered as the new information. This operation is known today as *AGM revision* and an extensive literature on the topic followed their original work.

Some years later, in 1991, Katsuno and Mendelzon [3] presented a new theory change operation which they called an *update*. In their paper, Katsuno and Mendelzon compared the update operation with the previous revision operation and provided some interesting remarks on the differences between the two approaches: while revision functions seemed well suited for modeling the change provoked by evolving knowledge about a static situation, update operations captured the change in knowledge provoked by an evolving situation.

[3] showed that the two operations have indeed different properties and, since then, AGM revision and update have been considered essentially different forms of theory change. The nature of their difference, though, is still an open question in the philosophical logic literature concerning theory change. For instance, are there other fundamentally different operations besides revision and update? The comparison of these two notions has been hampered by the fact that the two proposals do not stand on the same ground. While Katsuno and Mendelzon formalized their update operation as a binary connective between *formulas* in a (logically) finite language —specifically, a propositional language over a finite set of propositional variables—, Alchourrón, Gärdenfors and Makinson considered the general case of a possible infinite language and their revision operator takes *a theory and a formula* to the corresponding revised theory.

A number of formal distinctions between the two approaches has been already investigated, see [2], but the generalization of the update operation to infinite languages has not been previously attempted. After presenting briefly the standard update operation in Section 2, we define updates for infinite languages in Section 3. In Section 4 we prove that Katsuno and Mendelzon's original postulates characterizing finite updates are not sufficient for the infinite case. Finally in Section 5 we propose a strengthening of postulate (U8) which enables a representation theorem to be proved. In this way, our result puts revisions and updates in an even definitional basis that allows for a better comparison and understanding.

We will assume familiarity with Katsuno and Mendelzon's framework, as well as with basic notions in propositional logic. We consider a propositional language L and denote with P the set of all propositional letters. If P is finite we call L a finite (propositional) language. We take the consequence relation Cn on L to be supra-classical, compact and to satisfy the rule of introduction of disjunction into the premises; thus Cn satisfies the deduction theorem. Capital letters A, B are used to denote arbitrary formulae of L. A theory is a subset of L closed under Cn. Capital letters K, K_1, K_2 refer to theories of L, and we denote with $\mathbb{K}$ the class of all theories of L.

We take W as the set of all maximal consistent subsets of L. For the purposes of this work we consider the terms maximal consistent subset of L, possible world and valuation on L interchangeable. This, of course, amounts to working with models that are injective with respect to the interpretation function (no two distinct worlds satisfy exactly the same formulae) and full (every consistent set of formulae is satisfied by some world). The valuation function $[\ \ ] : L \to \mathcal{P}(W)$ is defined as usual, for any proposition letter p, $w \in [p]$ iff $p \in w$. Given $A \in L$ we denote by $[A]$ the proposition for A, i.e., the set of elements of W satisfying A. If K is a theory, $[K]$ denotes the set of possible worlds containing K. Given U a set of possible worlds, $\mathrm{Th}(U)$ returns the associated theory. A theory T is complete if for all $p \in P$ either $p \in T$ or $\neg p \in T$, and it is consistent if for no $p \in P$, both $p \in T$ and $\neg p \in T$.

A relation is a preorder if it is reflexive and transitive. It is an order if it is transitive and antisymmetric. A (pre)order relation R is total if for every $v, w \in W$, either $(w, v) \in R$ or $(v, w) \in R$. Otherwise R is a partial (pre)order. We will use the symbol $\preceq$ with subscripts for preorders. Whenever $w \preceq v$ but not $v \preceq w$ we will write $w \prec v$.

2 Katsuno and Mendelzon's update

Katsuno and Mendelzon define updates only for a classical propositional language based on a finite set of propositional variables P. This simplifying assumption has strong consequences as the set W of all possible valuations becomes finite. Two main properties result: every theory can be finitely axiomatized by a propositional formula; and every total order $\preceq$ on W is free of infinite descending chains. These two properties let Katsuno and Mendelzon provide a simple definition of the update operator as a binary connective $\diamond$ in the propositional language: $A \diamond B$ is a well formed formula denoting the result of updating the theory $\mathrm{Cn}(A)$ with the formula B. The $\diamond$ operator is characterized through the following postulates:

(u1) $A \diamond B$ implies B.

(u2) If A implies B then $A \diamond B$ is equivalent to A.

(u3) If both A and B are satisfiable then $A \diamond B$ is also satisfiable.

(u4) If A_1 is equivalent to A_2 and B_1 is equivalent to B_2 then $A_1 \diamond B_1$ is equivalent to $A_2 \diamond B_2$.

(u5) $(A \diamond B) \wedge C$ implies $A \diamond (B \wedge C)$.

(u6) If $A \diamond B_1$ implies B_2 and $A \diamond B_2$ implies B_1 then $A \diamond B_1$ is equivalent to $A \diamond B_2$.

(u7) If $\mathrm{Cn}(A)$ is complete then $(A \diamond B_1) \wedge (A \diamond B_2)$ implies $A \diamond (B_1 \vee B_2)$.

(u8) $(A_1 \vee A_2) \diamond B$ is equivalent to $(A_1 \diamond B) \vee (A_2 \diamond B)$.

They furthermore consider an additional postulate:

(u9) If $\mathrm{Cn}(A)$ is complete and $(A\Diamond B) \wedge C$ is satisfiable then $A\Diamond(B \wedge C)$ implies $(A\Diamond B) \wedge C$.

Katsuno and Mendelzon provide also a semantic characterization of the update operation through a notion of closeness between possible worlds. They consider a set of partial preorders on W, $\{\preceq_w : w \in W\}$. The intuitive meaning is that $v \preceq_w u$ if and only if v is at least as close to w as u is. The pointwise preorders $\preceq_w$ are then used in the definition of the update operation: given that any theory K can be semantically represented as a set of possible worlds $[K] = \{w_i \in W : K \subseteq w_i\}$, we can update K but considering the most plausible changes (according to $\preceq_{w_i}$) to each w_i to accommodate the new information. The only requirement on $\preceq_w$ is a *centering condition*, establishing that for every w, no world is as close to w as w itself: if $v \preceq_w w$ then $v = w$.

The following characterization results hold for the update operation, see [3] for the details.

Theorem 2.1 Let L be a finite propositional language. The update operator $\Diamond$ satisfies (u1)-(u8) iff there exists a model $\langle W, \{\preceq_w : w \in W\}\rangle$ where each $\preceq_w$ is a partial preorder over W that satisfies the centering condition, such that $[A\Diamond B] = \bigcup_{w\in[A]}\{v \in [B] : v \text{ is } \preceq_w\text{-minimal in } [B]\}$.

Theorem 2.2 Let L be a finite propositional language. The update operator $\Diamond$ satisfies (u1) - (u5),(u8) and (u9) iff there exists a model $\langle W, \{\preceq_w : w \in W\}\rangle$ where each $\preceq_w$ is a total preorder that satisfies the centering condition, such that $[A\Diamond B] = \bigcup_{w\in[A]}\{v \in [B] : v \text{ is } \preceq_w\text{-minimal in } [B]\}$. (Postulates (u6) and (u7) are superfluous in presence of the rest.)

3 Update for infinite languages

Following the notion of a change operator advocated by Alchourrón, Gärdenfors and Makinson [1], we generalize the update operation to theories. We redefine the update operator $\Diamond$ as a function that takes a theory and a formula and returns a theory, $\Diamond : I\!\!K \times L \rightarrow I\!\!K$. Notice that in a finite propositional language this is just a notational variant of Katsuno and Mendelzon's original setting.

It is quite straightforward to extend the characteristic pointwise semantics of the standard update function to infinite languages. The notion of closeness between worlds requires some adjustment. In addition to the centering condition, each $\preceq_w$ should satisfy what is known as the *limit assumption*: let A be any formula in L, then there exists some non-empty set Y, $Y \subseteq [A]$ such that each element in Y is a $\preceq_w$-minimal element of $[A]$. Formally,

$$\forall w \in W, \forall A \in L, \exists Y \subseteq [A], Y \neq \emptyset \text{ such that } \forall y \in Y, \forall x \in [A], y \preceq_w x.$$

Notice that the limit assumption is trivially satisfied in finite propositional languages.

Definition 3.1 (Update function) Let L be a possibly infinite propositional language. Let $\langle W, \{\preceq_w : w \in W\}\rangle$ be such that each $\preceq_w$ is a total preorder over W satisfying the centering condition and the limit assumption. We define $\blacklozenge : \mathbb{K} \times L \to \mathbb{K}$ as

$$K \blacklozenge A = \mathrm{Th}(\bigcup_{w \in [K]} \{v \in [A] : v \text{ is } \preceq_w \text{-minimal in } [A]\}).$$

We can straightforwardly recast the postulates governing the update function for possibly infinite languages as follows:

(U0) $K \diamondsuit A$ is a theory.

(U1) $A \in K \diamondsuit A$.

(U2) If $A \in K$ then $K \diamondsuit A = K$.

(U3) If $K \neq L$ and A is satisfiable then $K \diamondsuit A \neq L$.

(U4) If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$ then $K \diamondsuit A = K \diamondsuit B$.

(U5) $K \diamondsuit (A \wedge B) \subseteq \mathrm{Cn}(K \diamondsuit A \cup \{B\})$.

(U6) If $B \in K \diamondsuit A$ and $A \in K \diamondsuit B$ then $K \diamondsuit A = K \diamondsuit B$.

(U7) If K is a complete theory then $K \diamondsuit (A \vee B) \subseteq \mathrm{Cn}(K \diamondsuit A \cup K \diamondsuit B)$.

(U8) $(K_1 \cap K_2) \diamondsuit A = (K_1 \diamondsuit A) \cap (K_2 \diamondsuit A)$.

The additional postulate becomes:

(U9) If K is complete and $\mathrm{Cn}((K \diamondsuit A) \cup \{B\}) \neq L$ then $\mathrm{Cn}((K \diamondsuit A) \cup \{B\}) \subseteq K \diamondsuit (A \wedge B)$.

4 A non-representation theorem

The generalized version of the update postulates (U0)-(U9) do not characterize the update operation in a language with an infinite number of propositional letters.

Theorem 4.1 If L is an infinite propositional language, postulates (U0)-(U9) do not fully characterize the $\blacklozenge$ operation.

Proof. Given a propositional language L with an infinite but countable number of propositional letters we will exhibit a function $\circ : I\!K \times L \to I\!K$ satisfying (U0)-(U9) for which there is no model $\langle W, \{\preceq_w : w \in W\}\rangle$, satisfying that $\forall K \in I\!K$, $\forall A \in L, K \circ A = K \blacklozenge A$.

We semantically define $\circ$ as follows. Let us single out an (arbitrary) point v in W. For every $K \in I\!K$ and for every $A \in L$ define

$$[K \circ A] = \begin{cases} \emptyset & \text{if } [A] = \emptyset. \\ [K] & \text{if } [K] \subseteq [A]. \\ ([K] \cap [A]) \cup \{v\} & \text{if } A \in v \text{ and } [K] \cap [\neg A] \neq \emptyset \text{ is finite.} \\ [A] & \text{if } A \notin v \text{ or } [K] \cap [\neg A] \text{ is an infinite set.} \end{cases}$$

We first check that $\circ$ satisfies postulates (U0)-(U9). By definition $\circ$ trivially satisfies postulates (U0), (U1), (U2), (U3) and (U4).

(U5). We have to show that $K \circ (A \wedge B) \subseteq \mathrm{Cn}(K \circ A \cup \{B\})$ holds. There are three cases.

(a) If $[K] \subseteq [A]$ then $K \circ A = K$. If $\neg B \in K$, then $\mathrm{Cn}(K \circ A \cup \{B\}) = L$ and (U5) is verified. If $\neg B \notin K$, then $\mathrm{Cn}(K \circ A \cup \{B\}) = \mathrm{Cn}(K \cup \{B\})$. Since $A \in K$, $\mathrm{Cn}(K \cup \{B\}) = \mathrm{Cn}(K \cup \{A\} \cup \{B\}) = \mathrm{Cn}(K \cup \{A \wedge B\}) = K \circ (A \wedge B)$. Thus, (U5) holds.

(b) Assume $[K] \cap [\neg A] \neq \emptyset$ is a finite set. If $[K] \cap [\neg A \vee \neg B]$ is an infinite set or $A \wedge B \notin v$ then $K \circ (A \wedge B) = \mathrm{Cn}(A \wedge B)$ and (U5) holds. Suppose $[K] \cap [\neg A \vee \neg B]$ is finite and $A \wedge B \in v$. So $[K \circ (A \wedge B)] = ([K] \cap [A \wedge B]) \cup \{v\}$, while $[K \circ A] = ([K] \cap [A]) \cup \{v\}$. Since $B \in v$, $[K \circ A] \cap [B] = ((([K] \cap [A]) \cup \{v\}) \cap [B]) = ([K] \cap [A] \cap [B]) \cup (\{v\} \cap [B])$ $= ([K] \cap [A] \cap [B]) \cup \{v\} = [K \circ (A \wedge B)]$, thus (U5) is verified.

(c) If $[K] \cap [\neg A]$ is an infinite set then $[K] \cap [\neg A \vee \neg B]$ is also infinite. By definition $[K \circ (A \wedge B)] = [A \wedge B] = \mathrm{Cn}([A] \cup [B]) = \mathrm{Cn}(K \circ A \cup \{B\})$.

(U6). Suppose $B \in K \circ A$ and $A \in K \circ B$.

(a) If $[K] \subseteq [A]$ then $K \circ A = K$. Since $B \in K \circ A$, then $B \in K$, so $K \circ B = K = K \circ A$.

(b) Assume $[K] \cap [\neg A] \neq \emptyset$ is a finite set. If $A \in v$ then $[K \circ A] = ([K] \cap [A]) \cup \{v\}$. Since $B \in K \circ A$, then $([K] \cap [A]) \cup \{v\} \subseteq [B]$, and in particular, $B \in v$. Furthermore $[K] \cap [\neg B] \neq \emptyset$ is finite. Then, by definition, $[K \circ B] = ([K] \cap [B]) \cup \{v\}$. Since, in addition, $A \in K \circ B$, we obtain that $([K] \cap [B]) \cup \{v\} \subseteq [A]$. Therefore, $[K] \cap [A] = [K] \cap [B]$ and hence under the conditions in (b), $K \circ A = K \circ B$. Now suppose $A \notin v$. Then $[K \circ A]$ $= [A]$. Since $B \in K \circ A$, $[A] \subseteq [B]$. As $A \in K \circ B$, $[K \circ B] \subseteq [A]$. Hence $[K \circ B] \neq ([K] \cap [B]) \cup \{v\}$, because we assumed $A \notin v$. Hence, it must be that $[K \circ B] = [B]$, so $[B] \subseteq [A]$. Therefore, $[A] = [B]$ and $K \circ A = K \circ B$.

(c) Assume $[K] \cap [\neg A]$ is an infinite set. Then, $[K \circ A] = [A]$. Since $B \in K \circ A$, then $[A] \subseteq [B]$. There are two possibilities for $K \circ B$. If

$[K \circ B] = [B]$ then, using that $A \in K \circ B$, we obtain $[B] \subseteq [A]$ and $[K \circ A] = [K \circ B]$. If $[K \circ B] = ([K] \cap [B]) \cup \{v\}$ then $B \in v$ and $[K] \cap [\neg B]$ is a finite set. Because $A \in K \circ B$, $([K] \cap [B]) \cup \{v\} \subseteq [A]$, and $[K] \cap [B] \subseteq [K] \cap [A]$. Then, $[K] \cap [\neg A] \subseteq [K] \cap [\neg B]$; but this is impossible because we assumed $[K] \cap [\neg A]$ to be an infinite set and $[K] \cap [\neg B]$ to be finite.

(U7). We want to prove that if K is a complete theory then $K \circ (A \vee B) \subseteq \mathrm{Cn}(K \circ A \cup K \circ B)$. Assume K is complete.

If $A \in K$, $K \circ A = K$ and $K \circ (A \vee B) = K$. Thus, (U7) holds. If $\neg A \in K$, and $B \in K$, then $K \circ (A \vee B) = K \circ B = K$, so (U7) holds. If $\neg A \in K$, and $\neg B \in K$, if $A \in v$ or $B \in v$, then $K \circ (A \vee B) = v$, and either $K \circ B = v$ or $K \circ A = v$, so (U7) holds. If $\neg A \in v$ and $\neg B \in v$, then we obtain that $K \circ (A \vee B) = \mathrm{Cn}(A \vee B)$, $K \circ B = \mathrm{Cn}(B)$ and $K \circ A = \mathrm{Cn}(A)$. Hence, (U7) is verified.

(U8). We show that $(K_1 \cap K_2) \circ A = (K_1 \circ A) \cap (K_2 \circ A)$. Let $K = K_1 \cap K_2$.

(a) Assume $A \in K$. Then $K_1 \circ A = K_1$, $K_2 \circ A = K_2$ and $K \circ A = K$. Therefore (U8) is validated.

(b) Assume $[K] \cap [\neg A]$ is a finite non-empty set and $A \in v$. Then, $[K \circ A] = ([K] \cap [A]) \cup \{v\}$. If each $[K_i] \cap [\neg A]$, for $i = 1, 2$, is a finite set then $[K_i \circ A] = ([K_i] \cap [A]) \cup \{v\}$, $i = 1, 2$. So $[K \circ A] = ([K_1] \cap [A]) \cup ([K_2] \cap [A]) \cup \{v\} = [K_1 \circ A] \cup [K_2 \circ A]$. Otherwise, suppose $[K_1] \cap [\neg A]$ is an infinite set, and say $A \in K_2$. Then it also holds that $[K_1 \circ A] \cup [K_2 \circ A] = ((([K_1] \cap [A]) \cup \{v\}) \cup [K_2] = ((([K_1] \cap [A]) \cup \{v\}) \cup ([K_2] \cap [A]) = ([K_1] \cap [A]) \cup \{v\} \cup ([K_2] \cap [A]) = ((([K_1] \cup [K_2]) \cap [A]) \cup \{v\} = ([K] \cap [A]) \cup \{v\} = [K \circ A]$.

(c) Assume $[K] \cap [\neg A]$ is an infinite set or $\neg A \in v$. If $\neg A \in v$ then $K \circ A = K_1 \circ A = K_2 \circ A = \mathrm{Cn}(A)$, therefore, (U8) holds. Otherwise, either $[K_1] \cap [\neg A]$ or $[K_2] \cap [\neg A]$ or both are infinite sets. Clearly $[K \circ A] = [A]$ and, say, $[K_1] = [A]$. So $[K \circ A] = [K_1 \circ A]$, therefore, independently of the value of $[K_2 \circ A]$, we obtain that $[K \circ A] = [K_1 \circ A] \cup [K_2 \circ A]$.

(U9). Assume that K is complete and $[K \circ A] \cap [B] \neq \emptyset$. We prove that $[K \circ (A \wedge B)] \subseteq [K \circ A] \cap [B]$.

(a) If $A \in K$, $K \circ A = K$, by the hypotheses, $B \in K$. So $K \circ (A \wedge B) = K$. Thus, (U9) is verified.

(b) If $A \notin K$, then since K is complete $\neg A \in K$. If $A \in v$, $K \circ A = v$. By the hypothesis that $[K \circ A] \cap [B] \neq \emptyset$ we conclude $B \in v$. Thus, $[K \circ (A \wedge B)] \subseteq [K \circ A] \cap [B]$. In fact, $[K \circ (A \wedge B)] = [K \circ A] \cap [B] = \{v\}$. If $A \notin v$, $[K \circ A] = [A]$ and $[K \circ (A \wedge B)] = [A \wedge B]$. Thus, $[K \circ A] \cap [B] = [K \circ (A \wedge B)]$, hence (U9) is verified.

Now suppose for contradiction that there is a model $M = \langle W, \{\preceq_w : w \in W\} \rangle$, where each $\preceq_w$ is a total preorder on W satisfying the limit assumption and the centering condition, such that $\forall K \in \mathbb{K}$, $\forall A \in L, K \circ A = K \blacklozenge A$.

Thus, for every theory K such that $[K]$ is a finite set, and for every formula A, if $\neg A \in K$ and $A \in v$, where v is the distinguished point appearing in the definition of $\circ$ above, $K \circ A = K \blacklozenge A = v$ must hold. This translates into the following condition on the model M.

$$\forall x \in [\neg A], \forall y \in [A], v \neq y, \quad v \prec_x y.$$

Now let K be a theory such that $[K]$ is an infinite set and let $A \in L$ be such that $A \in v$ and $\neg A \in K$. Then by definition of $\circ$, $[K \circ A] = [A]$. However, $[K \blacklozenge A] = \bigcup_{x \in [K]} \{y \in [A] : y \text{ is } \preceq_x\text{-minimal in } [A]\} = \{v\}$. Because the language is infinite $\{v\} \neq [A]$. QED

5 A representation theorem

In the previous section we proved that postulates (U0)-(U9) are insufficient to characterize the update operation in an infinite language. We propose the following postulate as a strengthening of Katsuno and Mendelzon's postulate (U8) to achieve the representation result.

(IU8) If $K = \bigcap H_i$ then $K \diamondsuit A = \bigcap (H_i \diamondsuit A)$.

(IU8) states that the update of an intersection is the intersection of the updates. Obviously (IU8) implies (U8). We now prove that postulates (U0)-(U9) plus (IU8) completely characterize the update operation when infinite languages are allowed. In the proof we use the following result about consistent, complete theories.

Lemma 5.1 (Ventilation condition) Let $\diamondsuit$ be an update function satisfying postulates (U0)- (U9). If K is consistent and complete then for all $A, B \in L$, $K \diamondsuit A \vee B = K \diamondsuit A$ or $K \diamondsuit A \vee B = K \diamondsuit A$ or $K \diamondsuit A \vee B = K \diamondsuit A \cap K \diamondsuit B$.

Proof. Assume $K \diamondsuit A \vee B$ is different from $K \diamondsuit A$ and is also different from $K \diamondsuit B$. We want to prove that $K \diamondsuit A \vee B = K \diamondsuit A \cap K \diamondsuit B$. We will show the double inclusion.

($\supseteq$). This inclusion follows directly from (U5), which requires that $K \diamondsuit A \subseteq \mathrm{Cn}(K \diamondsuit A \vee B \cup \{A\})$ and $K \diamondsuit B \subseteq \mathrm{Cn}(K \diamondsuit A \vee B \cup \{B\})$. Then $K \diamondsuit A \cap K \diamondsuit B \subseteq \mathrm{Cn}(K \diamondsuit A \vee B \cup \{A\}) \cap \mathrm{Cn}(K \diamondsuit A \vee B \cup \{B\})$. By the rule of introduction of disjunction into the premises, $K \diamondsuit A \cap K \diamondsuit B \subseteq \mathrm{Cn}(K \diamondsuit A \vee B \cup \{A \vee B\}) = K \diamondsuit A \vee B$, using (U0) and (U1).

($\subseteq$). Suppose $\mathrm{Cn}(K \diamondsuit A \vee B \cup \{A\}) \neq L$ and $\mathrm{Cn}(K \diamondsuit A \vee B \cup \{B\}) \neq L$. By (U9) $\mathrm{Cn}(K \diamondsuit A \vee B \cup \{A\}) \subseteq K \diamondsuit A$ and $\mathrm{Cn}(K \diamondsuit A \vee B \cup \{B\}) \subseteq K \diamondsuit B$. Since $K \diamondsuit A \vee B \subseteq \mathrm{Cn}(K \diamondsuit A \vee B \cup \{A\}) \cap \mathrm{Cn}(K \diamondsuit A \vee B \cup \{B\})$, we have that $K \diamondsuit A \vee B \subseteq K \diamondsuit A \cap K \diamondsuit B$.

Now suppose $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) = L$ and $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) \neq L$ (the other is similar). Thus, $\neg B \in K\diamondsuit A \vee B$ and by (U1) $A \in K\diamondsuit A \vee B$. By (U6) If $A \in K\diamondsuit A \vee B$ and $A \vee B \in K\diamondsuit A$ then $K\diamondsuit A \vee B = K\diamondsuit A$, contradicting our initial assumption.

Finally, suppose $\mathrm{Cn}(K\diamondsuit A\vee B\cup\{A\}) = L$ and $\mathrm{Cn}(K\diamondsuit A\vee B\cup\{B\}) = L$. By (U1) $A \vee B \in K\diamondsuit A \vee B$. By $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) = L$, we have that $\neg A \in K\diamondsuit A \vee B$, thus $B \in K\diamondsuit A \vee B$. But $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) = L$, so $K\diamondsuit A \vee B = L$. Since K is consistent, by (U3) $A \vee B$ is unsatisfiable. Therefore A, B are both unsatisfiable formulae, and by (U1) $L = K\diamondsuit A\vee B = K\diamondsuit A = K\diamondsuit B$, again contradicting our initial assumptions. QED

Theorem 5.2 Let L be a possibly infinite propositional language, and let Cn be a classical consequence relation that is compact and satisfies the rule of introduction of disjunctions into the premises. An operator $\diamondsuit$ satisfies postulates (U0)-(U7), (IU8), (U9) if and only if there exists a model $M = \langle W, \{\preceq_w\colon w \in W\}\rangle$, where each $\preceq_w$ is a total preorder over W centered in w that satisfies the limit assumption and for any $K \in \mathbb{K}$, $A \in L$, $K\diamondsuit A = K\blacklozenge A$.

Proof.
[$\Leftarrow$]. We have to show that the operator $\blacklozenge$ satisfies postulates (U0)-(U7), (IU8) and (U9).

(U0) and (U1). Granted since, by Definition 3.1, $[K\blacklozenge A] \subseteq [A]$.

(U2). Follows as a consequence of the centering condition.

(U3). Follows by the definition of min on nonempty sets.

(U4). Obvious from the semantic definition of the update operation.

(U5). We have to show that $[K\blacklozenge A]\cap[B] \subseteq [K\blacklozenge(A\wedge B)]$. If $[K\blacklozenge A]\cap[B] = \emptyset$, the inclusion trivially holds. Assume $[K\blacklozenge A] \cap [B] \neq \emptyset$. Let u be any in $[K\blacklozenge A] \cap [B]$. Then $u \in \bigcup_{w\in[K]}\{v \in [A] : v \text{ is } \preceq_w\text{-minimal in } [A]\} \cap [B] = \bigcup_{w\in[K]}\{v \in [A] \cap [B] : v \text{ is } \preceq_w\text{-minimal in } [A]\}$. Let $w_0 \in [K]$ be such that u is $\preceq_{w_0}$-minimal in $[A]$. That is $\forall v \in [A]$, $u \preceq_{w_0} v$. A fortiori, $u \in [A]\cap[B]$. Thus, there is no $v \in [A]\cap[B]$ such that $v \prec_{w_0} u$, so u is indeed $\preceq_w$-minimal in $[A] \cap [B]$.

(U6). Assume $B \in K\blacklozenge A$ and $A \in K\blacklozenge B$. We want to show $[K\blacklozenge A] = [K\blacklozenge B]$. $[K\blacklozenge A] = \bigcup_{w\in[K]}\{v \in [A] : v \text{ is } \preceq_w\text{-minimal in } [A]\}$. By the hypothesis that $B \in K\blacklozenge A$, $[K\blacklozenge A] \subseteq [B]$. So, $[K\blacklozenge A] = \bigcup_{w\in[K]}\{v \in [A] \cap [B] : v \text{ is } \preceq_w \text{-minimal in } [A] \cap [B]\}$. Similarly, $[K\blacklozenge B] = \bigcup_{w\in[K]}\{v \in [B] : v \text{ is } \preceq_w\text{-minimal in } [B]\}$, and by the hypothesis that $A \in K\blacklozenge B$, $[K\blacklozenge B] \subseteq [A]$. Hence, $[K\blacklozenge B] = \bigcup_{w\in[K]}\{v \in [A] \cap [B] : v \text{ is } \preceq_w\text{-minimal in } [A] \cap [B]\}$. Therefore, $[K\blacklozenge A] = [K\blacklozenge B]$, as required.

(U7). We have to prove that when $[K]$ is a singleton $[K\blacklozenge A] \cap [K\blacklozenge B] \subseteq [K\blacklozenge(A\vee B)]$. Assume $[K] = \{u\}$. Then, $[K\blacklozenge A] = \{v \in [A] : v \text{ is } \preceq_u\text{-minimal}$

in $[A]\}$, while $[K\blacklozenge B] = \{v \in [B] : v$ is $\preceq_u$-minimal in $[B]\}$. Furthermore $[K\blacklozenge(A \vee B)] = \{v \in [A \vee B] : v$ is $\preceq_u$-minimal in $[A \vee B]\} = \{v \in [A] \cup [B] : v$ is $\preceq_u$-minimal in $[A] \cup [B]\} = \{v \in [A] \cup [B] : v$ is $\preceq_u$-minimal in $[A]$ or v is $\preceq_u$-minimal in $[B]\}$. And finally, $[K\blacklozenge A] \cap [K\blacklozenge B] = \{v \in [A] \cap [B] : v$ is $\preceq_u$ -minimal in $[A]$ and v is $\preceq_u$ -minimal in $[B]\}$. Thus, $[K\blacklozenge A] \cap [K\blacklozenge B] \subseteq [K\blacklozenge(A \vee B)]$.

(IU8). Assume $[K] = \bigcup_{i\in I}[K_i]$ to show $[K\blacklozenge A] = \bigcup_{i\in I}[K_i\blacklozenge A]$. By definition, $[K\blacklozenge A] = \bigcup_{w\in\bigcup_{i\in I}[K_i]}\{v \in [A] : v$ is $\preceq_w$-minimal in $[A]\} = \bigcup_{i\in I}(\bigcup_{w\in[K_i]}\{v \in [A] : v$ is $\preceq_w$ -minimal in $[A]\}) = \bigcup_{i\in I}[K_i\blacklozenge A]$.

(U9). Assume $[K] = \{u\}$ and $([K\blacklozenge A]) \cap [B] \neq \emptyset$. We have to show $[K\blacklozenge(A \wedge B)] \subseteq [K\blacklozenge A] \cap [B]$. Suppose there is some $y \in [K\blacklozenge A \wedge B]$ but $y \notin [K\blacklozenge A] \cap [B]$. Then $[K\blacklozenge A] \subseteq [\neg B]$, contradicting $[K\blacklozenge A] \cap [B] \neq \emptyset$.

$[\Rightarrow]$. Let $\diamondsuit$ be a change function satisfying (U0)-(U7), (IU8) and (U9). We will construct a model $M = \langle W, \{\preceq_w : w \in W\}\rangle$ such that for every theory $K \in I\!K$ and formula $A \in L$, $K\diamondsuit A = K\blacklozenge A$.

We start by defining the model M. The domain W will be the set of all complete consistent theories in the language L. Assume $\{\preceq_w : w \in W\}$ is the set of relations defined by:

(i.) $v \preceq_w u$ iff there exists $A \in v \cap u$ such that $v \in [w\diamondsuit A]$ or there exists no satisfiable A such that $u \in [w\diamondsuit A]$.

We will show that M is an update model by demonstrating that each $\preceq_w$ is a total preorder satisfying the centering condition and the limit assumption.

(a) $\preceq_w$ is *totally connected*. Suppose $u \not\preceq_w v$ and $v \not\preceq_w u$. Then for some consistent $A, B \in L$, $v \in [w\diamondsuit A]$ and $u \in [w\diamondsuit B]$. Then, by Lemma 5.1 we have that $w\diamondsuit A\vee B = w\diamondsuit A$ or $w\diamondsuit A\vee B = w\diamondsuit B$ or $w\diamondsuit A\vee B = w\diamondsuit A\cap w\diamondsuit B$. Thus one of v or u is in $[w\diamondsuit A\vee B]$ contradicting the fact that neither $v \preceq_w u$ nor $u \preceq_w v$. (Notice that total connectedness implies reflexivity).

(b) $\preceq_w$ is *transitive*. Suppose $u \preceq_w v$ and $v \preceq_w z$. If there is no satisfiable A such that $z \in [w\diamondsuit A]$ then also $u \preceq_w z$ and we are done. Otherwise, $v \preceq_w z$ because there exists $C \in u \cap v$ such that $u \in [w\diamondsuit C]$, but then also there exists $B \in v \cap z$ such that $v \in [w\diamondsuit B]$. Now $\neg C \notin K\diamondsuit B$, so by Lemma 5.1 $\neg C \notin [w\diamondsuit B \vee C]$. This means that $w\diamondsuit((B \vee C) \wedge C)$ $=w\diamondsuit(B \vee C) \cup \{C\}$ $= K\diamondsuit C$. Namely $[w\diamondsuit C] = [w\diamondsuit B \vee C] \cap [C]$. Then $[K\diamondsuit C] \subseteq [K\diamondsuit(B \vee C)]$ and $u \in [K\diamondsuit(B \vee C)]$. But $B \vee C \in z \cap u$, so $u \preceq_w z$.

(c) $\preceq_w$ is *centered*. Suppose $v \neq w$ and $v \preceq_w w$. Trivially, from the postulates, $w \in [w\diamondsuit\top]$, hence by definition (i.) $v \preceq_w w$ implies there is some $A \in v \cap w$ such that $v \in [w\diamondsuit A]$. But this contradicts postulate (U2) which requires $[w\diamondsuit A] = \{w\}$.

(d) That $\preceq_w$ satisfies the *limit assumption* follows directly from postulate (U3), which implies that for every satisfiable A, and for every $w \in W$, $[w\Diamond A]$ must be non empty. Then there must be some $v \in [A]$ that is minimal in $\preceq_w$ such that $v \in [w\Diamond A]$.

It remains to show that the update function determined by M is $\Diamond$. In the limiting case when K is the inconsistent theory or A is unsatisfiable, $K\Diamond A$ and $K\blacklozenge A$ agree. We will now prove, for K and A satisfiable, that $u \in [K\Diamond A]$ iff $u \in [K\blacklozenge A]$ by analyzing the different cases.

Suppose $[K] = \{w\}$.

$[K\Diamond A] \subseteq [K\blacklozenge A]$. Let $v \in [K\Diamond A]$. By postulate (U1), $[K\Diamond A] \subseteq [A]$, so $v \in [A]$. By (i.), for every $u \in [A]$, $v \preceq_w u$. Hence, $v \in \{y \in [A] : y \text{ is } \preceq_w\text{-minimal in } [A]\} = [K\blacklozenge A]$.

$[K\blacklozenge A] \subseteq [K\Diamond A]$. Let $v \in [K\blacklozenge A]$. By definition of $\blacklozenge$, $v \in \{y \in [A] : y \text{ is } \preceq_w\text{-minimal in } [A]\}$. So for all $u \in [A]$, $v \preceq_w u$; thus, by (i.), $v \in [w\Diamond A]$.

The general case, $[K] > 1$.

$[K\Diamond A] \subseteq [K\blacklozenge A]$. Let $v \in [K\Diamond A]$. By postulate (IU8), if $[K] = \bigcup_{i\in I}[K_i]$ then $[K\Diamond A] = \bigcup_{i\in I}[K_i\Diamond A]$.

In particular, $[K] = \bigcup_{i\in I}[T_i]$ for complete theories T_i. Thus, $v \in \bigcup_{i\in I}[T_i\Diamond A]$. Hence, v must be in, say, some $[T_j\Diamond A]$, $j \in I$. Then, by the previous case, $v \in [T_j\blacklozenge A]$. Therefore, $v \in \bigcup_{w\in[K]}\{y \in [A] : y \text{ is } \preceq_w \text{-minimal in } [A]\} = [K\blacklozenge A]$.

$[K\blacklozenge A] \subseteq [K\Diamond A]$. Let $v \in [K\blacklozenge A]$. Then, $v \in \bigcup_{w\in[K]}\{y \in [A] : y \text{ is } \preceq_w\text{-minimal in } [A]\}$. In particular, there exists some $w \in [K]$ such that $v \in \{y \in [A] : y \text{ is } \preceq_w \text{-minimal in } [A]\}$. By the previous case, $v \in [w\Diamond A]$. But $[K] = \bigcup_{i\in I}[T_i]$ for complete theories T_i, such that $w = T_j$, for some $j \in I$. By postulate (IU8) we obtain that when $[K] = \bigcup_{i\in I}[K_i]$, $[K\Diamond A] = [T_j\Diamond A] \cup (\bigcup_{i\in I, i\neq j}[T_i\Diamond A])$. Hence $v \in [K\Diamond A]$. QED

Katsuno and Mendelzon's characterization results based on partial orders as opposed to partial pre-orders also lift to the infinite case, replacing postulate (U8) with postulate (IU8).

6 Conclusions

In the present paper we have shown a generalization of Katsuno and Mendelzon's update to propositional languages over a possible infinite set of propositional letters. We prove that the original update postulates are not sufficient to characterize update functions on infinite languages. Furthermore we provide a strengthening of the axiomatization which leads to a representation theorem (Theorem 5.2).

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