Modal Logics with Counting

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Abstract. We present a modal language that includes explicit operators to count the number of elements that a model might include in the extension of a formula, and we discuss how this logic has been previously investigated under different guises. We show that the language is related to graded modalities and to hybrid logics. We illustrate a possible application of the language to the treatment of plural objects and queries in natural language. We investigate the expressive power of this logic via bisimulations, discuss the complexity of its satisfiability problem, define a new reasoning task that retrieves the cardinality bound of the extension of a given input formula, and provide an algorithm to solve it.

1 Counting, Modally

Suppose there are at least two apples (say, on the table, but we don't care at the moment where the apples are). First-order logic ($\mathcal{FOL}$) with equality has no problem expressing this fact[1]:

$$\exists x.\exists y.(x \neq y \wedge Apple(x) \wedge Apple(y)).$$

We can actually dispense with equality, if we introduce counting quantifiers [1]

$$\exists^{\geq 2} x.Apple(x).$$

But suppose that we want to dispense with *quantifiers* instead, and count in terms of a propositional (or a modal) language. The following representation seems quite natural (arguably, even more natural than the first-order counterparts with or without counting quantifiers)

$$Apple \geq 2.$$

In this paper we will investigate propositional and modal languages extended with such counting operators. Let us be bold and introduce, already, the formal syntax and semantics of the basic modal logic with counting $\mathcal{MLC}$, the main language we want to explore:

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[1] It is well known that $\mathcal{FOL}$ can express any finite counting quantifier.

Definition 1 (Syntax). *Let* $\mathsf{Prop} = \{p_1, p_2, \dots\}$ *(the* propositional symbols*) and* $\mathsf{Rel} = \{r_1, r_2, \dots\}$ *(the* relational symbols*) be disjoint, countable infinite sets. The set* Forms *of formulas of* $\mathcal{MLC}$ *over signature* $\langle \mathsf{Prop}, \mathsf{Rel} \rangle$ *is defined as:*

$$\mathsf{Forms} ::= \bot \mid p \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \langle r \rangle \varphi \mid (\varphi \geq n) \mid (\varphi \leq n),$$

for $p \in \mathsf{Prop}$*,* $r \in \mathsf{Rel}$*,* $\varphi, \varphi_1, \varphi_2 \in \mathsf{Forms}$ *and* n *a natural number. Other Boolean and modal operators are defined as usual, and we define* $(\varphi = n)$ *as* $(\varphi \geq n) \wedge (\varphi \leq n)$*,* $(\varphi > n)$ *as* $(\varphi \geq (n+1))$ *and* $(\varphi < n)$ *as* $(\varphi \leq (n-1))$ *if* $n > 0$ *or* $\bot$ *otherwise.*

We will call $\mathcal{PLC}$ the "propositional fragment," i.e., the fragment obtained by dropping $\langle r \rangle \varphi$. Let us now introduce the semantics.

Definition 2 (Semantics). *Given a signature* $\mathcal{S} = \langle \mathsf{Prop}, \mathsf{Rel} \rangle$*, a* model *for* $\mathcal{S}$ *is a tuple* $\langle W, (R_r)_{r \in \mathsf{Rel}}, V \rangle$*, satisfying the following conditions: (i)* $W \neq \emptyset$ *(elements in* W *are called states); (ii) each* R_r *is a binary relation on* W *(usually called accessibility relations); (iii)* $V : \mathsf{Prop} \to 2^W$ *is a labeling function.*

Given the model $\mathcal{M} = \langle W, (R_r)_{r \in \mathsf{Rel}}, V \rangle$ *and* $w \in W$*, the semantics for the different operators is defined as follows:*

$$\begin{array}{lcl}
\mathcal{M}, w \models p & \iff & w \in V(p), \quad p \in \mathsf{Prop} \\
\mathcal{M}, w \models \neg\varphi & \iff & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \iff & \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \langle r \rangle \varphi & \iff & \text{there is } w' \text{ such that } R_r(w, w') \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models (\varphi \geq n) & \iff & |\{w \mid \mathcal{M}, w \models \varphi\}| \geq n \\
\mathcal{M}, w \models (\varphi \leq n) & \iff & |\{w \mid \mathcal{M}, w \models \varphi\}| \leq n.
\end{array}$$

We will say that a formula φ *is* satisfiable*, if there is a model* $\mathcal{M}$ *and a state* w *in its domain such that* $\mathcal{M}, w \models \varphi$*. For a set of formulas* $\Gamma \cup \{\varphi\}$ *we say that* $\Gamma \models \varphi$ *if and only if for any model* $\mathcal{M}$ *and any* w *in its domain* $\mathcal{M}, w \models \Gamma$ *implies* $\mathcal{M}, w \models \varphi$ *(this relation is sometimes called* local entailment*). The* extension $\|\varphi\|^{\mathcal{M}}$ *of a formula* φ *in a model* $\mathcal{M}$ *is the set* $\{w \mid \mathcal{M}, w \models \varphi\}$*, and the* theory *of* w *in* $\mathcal{M}$*, notation* $\mathsf{Th}^{\mathcal{M}}(w)$*, is the set* $\{\varphi \mid \mathcal{M}, w \models \varphi\}$*. When the model* $\mathcal{M}$ *is clear from context we will drop the super-indexes. We will write* $\mathcal{M}, w \equiv_{\mathcal{MLC}} \mathcal{M}', w'$ *if* $\mathsf{Th}^{\mathcal{M}}(w) = \mathsf{Th}^{\mathcal{M}'}(w')$*.*

It should be clear from Definitions 1 and 2 that $\mathcal{MLC}$ is indeed the basic modal logic $\mathcal{ML}$ [2] extended with the counting operators. We will be mainly discussing extensions of $\mathcal{ML}$ for simplicity. We could have naturally added the counting operators to any modal logic, e.g., temporal logic with counting.

The $\mathcal{MLC}$ language and, in particular, its sublanguage $\mathcal{PLC}$ have been investigated under different guises. $\mathcal{PLC}$ is introduced as the logic $S5_n$ by Fine in [3] where the, by now well studied, notion of *graded modalities* was introduced. The semantic definition of the graded modality $\langle r \rangle_n \varphi$ is given by the condition

$$\mathcal{M}, w \models \langle r \rangle_n \varphi \iff |\{w' \mid R_r(w, w') \text{ and } \mathcal{M}, w' \models \varphi\}| \geq n.$$

$S5_n$ is the logic obtained when the $\langle r \rangle_n$ operator is restricted to models where R_r is interpreted as an equivalence relation. Now, if R_r is the universal relation, then $\langle r\rangle_n\varphi$ is trivially equivalent to $(\varphi \geq n)$. But a well known result (see, e.g. [2]) establishes that the modal logic of the universal relation coincides with the modal logic obtained when we only require the accessibility relation to be an equivalence relation. The main contribution of [3] is to provide sound and complete axiomatizations for these languages. The original results of Fine were extended by van der Hoek and de Rijke in [4, 5]. In addition to providing further axiomatizations, investigating normal forms, and establishing the complexity of the satisfiability problem for different logics with graded modalities, the authors propose these languages as a modal framework where some ideas from the Theory of Generalized Quantifiers [6] could be investigated by means of modal tools.

The relation between $\mathcal{MLC}$ and graded modalities was also discovered in the field of description logics. In this area, graded modalities are called *cardinality restrictions* and Baader *et al.* investigate in [7] *concept cardinality restrictions* which coincide exactly with the counting operators we defined. Interestingly, they decide to add concept cardinality restrictions not as operators of the concept language, but as a more expressive kind of terminological axioms, and they remark that they can express classical terminological axioms of the form $\varphi \sqsubseteq \psi$. $\varphi \sqsubseteq \psi$ is satisfied in the model if the interpretation of φ is a subset of the interpretation of ψ, and indeed this is the case exactly when $((\varphi\wedge\neg\psi) \leq 0)$. The main contribution of [7] is the definition of sound, complete and terminating tableaux calculus for these languages. A detailed complexity analysis of the satisfiability problem for the language and an optimal tableaux calculus is given in [8].

Another way of explaining why counting operators can express terminological axioms is realizing that they can express the universal modality $\mathsf{A}\varphi$ [9]:

$$\mathcal{M}, w \models \mathsf{A}\varphi \iff \text{for all } w', \mathcal{M}, w' \models \varphi.$$

$\mathsf{A}\varphi$ is equivalent to $((\neg\varphi) \leq 0)$, and $\varphi \sqsubseteq \psi$ is equivalent to $\mathsf{A}(\varphi \to \psi)$. Actually, counting modalities can also express *nominals* (i.e., special propositional symbol whose interpretations are restricted to singleton subsets of the domain) by just stating $(p = 1)$ for p a propositional symbol, and hence they can be considered also as hybrid logics [10].

In this article, we provide new results about the $\mathcal{MLC}$ language. Our first contribution is conceptual, rather than technical, and it can be simple put as follows. The counting operators $(\varphi \geq n)$ and $(\varphi \leq n)$ are interesting on their own, independently of their relation with graded modalities. They are *global* operators (with a behavior similar to the universal modality or satisfiability operators), and they can be naturally combined with local operators (as is commonly done in hybrid languages). They are also modular, and they can naturally be added to any modal language. In a slogan: counting operators are the modal counterpart of first-order counting quantifiers.

In Section 2 we show how $\mathcal{MLC}$ can be used as representation language in a natural language application modeling queries including plurals. In Section 3 we will investigate the expressive power of $\mathcal{MLC}$ using a suitable notion of bisimulation. In Section 4 we first discuss the complexity of the satisfiability problem, drawing from previously known results, we then introduce a new reasoning task and devise an algorithm to solve it.

2 Representing Plurals in Natural Language

We discuss here a possible representation of plurals and references in $\mathcal{MLC}$, intended to be used in natural language processing tasks such as reference resolution or generation as is done in, e.g., [11]. The idea is to represent the information introduced in a discourse as a set of $\mathcal{MLC}$ formulas $\varGamma$, and being able to express and answer queries of the form "how many of a certain kind of objects are there?" in this context[2].

As we saw in the previous section, $\mathcal{MLC}$ enables us to assert the cardinality of a proposition in the model. For example, $\varGamma = \{(\mathit{Apple} \wedge \mathit{Red}) = 2\}$ represents "there are two red apples", and the query "how many $(\mathit{Apple} \wedge \mathit{Red})$?" should return "2". But suppose that we want to *refer* to "two red apples" (i.e., we don't know how many red apples are there in total, but we want to refer to two of them). For the representation of this kind of reference we need to be able to *name* the referred group of object by, for example, introducing a new propositional symbol a_1 and adding to $\varGamma$ the formula[3]:

$$\text{"two red apples": } (a_1 = 2) \wedge (a_1 \sqsubseteq (\mathit{Apple} \wedge \mathit{Red}))$$

In this case, a query "how many $(\mathit{Apple} \wedge \mathit{Red})$?" cannot be answered (i.e., is undefined) since the total number of apples in the model is not known. But the query "how many a_1?" should return "2."

If now we add that there are also two green apples and want to refer to that group, we need to introduce another propositional symbol a_2 and add to $\varGamma$:

$$\text{"two green apples": } (a_2 = 2) \wedge (a_2 \sqsubseteq (\mathit{Apple} \wedge \mathit{Green}))$$

Now, the number of apples that are in the group formed by a_1 and a_2 (i.e., $a_1 \vee a_2$) is also undefined because nothing prevents those two sets from overlapping. If we explicitly say that the group are disjoint $(a_1 \sqsubseteq \neg a_2)$ or that the colors are mutually exclusive $(\mathit{Green} \sqsubseteq \neg \mathit{Red})$ for that we should be able to answer "4".

Suppose that now we learn that "three of the apples are rotten." This reference creates a new group containing all the apples mentioned up to now:

$$(a_3 \sqsubseteq (a_1 \vee a_2)) \;\wedge\; ((a_1 \vee a_2) \sqsubseteq a_3)$$

And then assert that three of them are rotten by adding to $\varGamma$ $(a_3 \wedge \mathit{Rotten}) = 3)$. If we further discover that all the red apples are rotten $(a_1 \sqsubseteq \mathit{Rotten})$, querying for "how many green apples are rotten," i.e., "how many $(a_2 \wedge \mathit{Rotten})$" will returns "1".

In Section 4 we introduce the inference task of counting that corresponds to the finite cardinality queries we just discussed. But first, in the next section, we investigate in detail the expressive power of $\mathcal{MLC}$.

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[2] This representation does not aim to solve all the issues concerning the use of plurals in natural language (e.g., the distributive versus collective readings of certain adjectives when applied to sets of objects), which are known to be difficult to model [12]. For further details see, for example, [13].

[3] Remember that $\varphi \sqsubseteq \psi$ as a short hand for $\mathsf{A}(\varphi \rightarrow \psi)$ or, equivalently, $(\varphi \wedge \neg\psi) \leq 0$.

3 The Expressive Power of $\mathcal{MLC}$

To get more familiar with the language, let us start with some examples of what can be expressed in $\mathcal{MLC}$. We can, for example, fix the size of the model to any finite cardinality by setting

$$(\top = n)$$

for n a natural number. The formula also shows that, if numbers are coded in binary, then neither $\mathcal{MLC}$ nor $\mathcal{PLC}$ has the polysize model property.

Proposition 1. *If numbers are coded in binary, then there are formulas in $\mathcal{PLC}$ (and hence also in $\mathcal{MLC}$) whose only models are exponentially larger.*

Notice that counting operators can be nested. For example $((p \geq 1) \geq 1)$ is a well formed formula, which it is actually equivalent to $(p \geq 1)$. But, as it is discussed in [4], every formula in $\mathcal{MLC}$ is equivalent to a formula where each counting operators appears under the scope of neither modal nor counting operators. The proof uses the fact that for any counting subformula σ appearing in a formula φ we have that the following is valid

$$\varphi[\sigma] \leftrightarrow (\sigma \rightarrow \varphi[\sigma/\top]) \wedge (\neg\sigma \rightarrow \varphi[\sigma/\bot])$$

Other operators with a global semantics, like the universal modality A or satisfiability operators $@_i$, have the same property. Notice though, that the formula we obtain after extracting all counting operators can be exponentially larger. If we only require equi-satisfiability (and not equivalence), we can use the method of [14] to obtain a formula which is only polynomially larger. We will return to this issue in Section 4.

As we mentioned in the introduction, the hybrid logic $\mathcal{H}(\mathsf{A})$ (the basic modal logic extended with nominals and the universal modality [10]) is a sublogic of $\mathcal{MLC}$, as the language can express nominals and the universal modality. It can even express the difference modality $\mathsf{D}\varphi$ [15] with semantics

$$\mathcal{M}, w \models \mathsf{D}\varphi \iff \text{there is } w' \neq w \text{ and } \mathcal{M}, w' \models \varphi$$

as $\mathsf{D}\varphi$ is equivalent to $(\varphi \rightarrow (\varphi \geq 2)) \wedge (\neg\varphi \rightarrow (\varphi \geq 1))$. On the other hand, the expressive power of counting and graded modalities is incomparable. We will establish this in Theorem 3 using a suitable notion of bisimulation for $\mathcal{MLC}$ that we now introduce

Definition 3 (Bisimulation). *A* bisimulation *between two models $\mathcal{M} = \langle W, (R_r)_{r\in\mathsf{Rel}}, V\rangle$ and $\mathcal{M}' = \langle W', (R'_r)_{r\in\mathsf{Rel}}, V'\rangle$ is a non-empty binary relation E between their domains (that is, $E \subseteq W \times W'$) such that whenever wEw' we have:*

Atomic harmony: *w and w' satisfy the same propositional symbols.*
Zig: *if $R_r wv$ then there exists a point $v' \in W'$ such that vEv' and $R'_r w'v'$.*
Zag: *if $R'_r w'v'$ then there exists a point $v \in W$ such that vEv' and $R_r wv$.*
Bijectivity: *E contains a bijection between W and W'.*

For two models $\mathcal{M}$ and $\mathcal{M}'$ and two elements w and w' in their respective domains, we write $\mathcal{M}, w' \leftrightarroweq \mathcal{M}', w'$ if there exists a bisimulation between $\mathcal{M}, w'$ and $\mathcal{M}', w'$ linking w and w'.

Theorem 1. *If $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ then $\mathcal{M}, w$ and $\mathcal{M}', w'$ satisfy the same formulas of $\mathcal{MLC}$.*

Proof. Assume there is a bisimulation E between $\mathcal{M}$ and $\mathcal{M}'$. Because of Atomic harmony, Zig and Zag, we now that E preserves all formulas of the basic modal language [2]. We only need to consider the counting operators.

Suppose then that $\varphi = (\psi \geq n)$ and let f be one bijection that by definition is contained in the bisimulation linking $\mathcal{M}$ and $\mathcal{M}'$. Assume that $\mathcal{M}, w \models (\psi \geq n)$. By inductive hypothesis $f(\|\psi\|^{\mathcal{M}}) \subseteq \|\psi^{\mathcal{M}'}\|$ and because f is a injective $|f(\|\psi\|^{\mathcal{M}})| \geq n$, hence $\mathcal{M}', w' \models (\psi \geq n)$. For the other direction, assume $\mathcal{M}', w' \models (\psi \geq n)$. Because f is a bijection we can consider $f^{-1}(\|\psi\|^{\mathcal{M}'})$ which has size greater than n , and by inductive hypothesis we know that it is a subset of $\|\psi\|^{\mathcal{M}}$. Hence $\mathcal{M}, w \models (\psi \geq n)$. The case for $\varphi = (\psi \leq n)$ is similar.

As usual, the converse is not necessarily true but it holds on finite models.

Theorem 2. *Let $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ be two finite models and $(w, w') \in W \times W'$, $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ if and only if $\mathcal{M}, w \equiv_{\mathcal{MLC}} \mathcal{M}', w'$.*

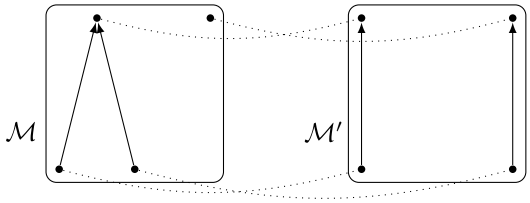
Proof. The implication from left to right is given by Theorem 1. For the other implication, we have to prove that $\equiv_{\mathcal{MLC}}$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ that links w and w'. Atomic harmony, Zig and Zag are proved in the standard way (see [2]). To prove that $\equiv_{\mathcal{MLC}}$ contains a bijection reason as follows.

Consider every pair of subsets (C, C'), $C \subseteq W$, $C' \subseteq W'$ such that for all $(a, b) \in C \times C'$, $\mathcal{M}, a \equiv_{\mathcal{MLC}} \mathcal{M}', b$. There is at least one such pair by hypothesis. Enumerate these pairs as $(C_1, C'_1), \ldots, (C_n, C'_n)$ (as the model is finite there is only a finite number of them), and let $\Sigma_1, \ldots, \Sigma_n$ be such that $\Sigma_i = \mathsf{Th}(a)$ for some $a \in C_i \cup C'_i$ (by construction all elements in $C_i \cup C'_i$ satisfy the same formulas of $\mathcal{MLC}$). Now choose for each i, $\varphi_i \in \Sigma_i$ such that for all $j \neq i$, $\varphi_i \notin \Sigma_j$. Notice that $|C_i| = |\|\varphi_i\|^{\mathcal{M}}|$ and that $|C'_i| = |\|\varphi_i\|^{\mathcal{M}'}|$, we want to prove that $|C_i| = |C'_i|$. But by hypothesis $\mathcal{M}, w \equiv_{\mathcal{MLC}} \mathcal{M}', w'$, and then $\mathcal{M}, w \models \varphi_i = n$ if and only if $\mathcal{M}', w' \models \varphi_i = n$.

As C_i and C'_i have the same cardinality we can define an injective function $f : \bigcup C_i \to \bigcup C'_i$, such that for $a \in C_i$, $f(a) \in C'_i$. It only rests to prove that f is total and surjective.

Suppose there is $a \in W$ such that $a \notin \bigcup C_i$, then there is no element a' in W' such that $\mathcal{M}, a \equiv_{\mathcal{MLC}} \mathcal{M}', a'$. For each $a'_i \in W'$, let φ_i be a formula such that $\varphi_i \in \mathsf{Th}(a)$ but $\varphi_i \notin \mathsf{Th}(a')$. But then $\mathcal{M}, w \models (\bigwedge \varphi_i \geq 1)$ while $\mathcal{M}, w' \not\models (\bigwedge \varphi_i \geq 1)$ contradicting hypothesis. In a similar way we can prove that f is surjective.

Notice that $\mathcal{MLC}$-bisimulations are not isomorphisms. The following two models, for example, are $\mathcal{MLC}$-bisimilar but not isomorphic.

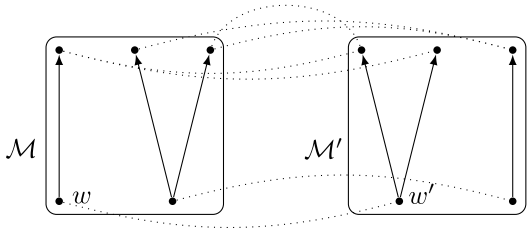


$\mathcal{M}$ and $\mathcal{M}'$ can be differentiated by the first order sentences $\exists x.\forall y.(\neg R(x,y) \wedge \neg R(y,x))$. But there is no $\mathcal{MLC}$ formula which is globally true in one model but false in the other. On the other hand, [6] proves that every sentence of first-order logic with equality and only monadic propositional symbols is equivalent to the translation of a formula in $\mathcal{PLC}$.

We now return to the comparison of $\mathcal{MLC}$ and graded modalities.

Theorem 3. *The expressive power of counting modalities and graded modalities is incomparable (when interpreted on the set of all possible models).*

Proof. Consider the following two models $\mathcal{M}$ and $\mathcal{M}'$. It is not difficult to verify that the dotted arrows defines a $\mathcal{MLC}$-bisimulation.



$\mathcal{M}, w \not\models \langle r \rangle_2 \top$ while $\mathcal{M}', w' \models \langle r \rangle_2 \top$ while no formula of $\mathcal{MLC}$ can differentiate w and w'[4]. For the other direction, just consider a model with one state and another model with two states. Clearly, the models cannot be distinguished using graded modalities (as they can only count the number of successors) but the counting forma formula $(\top \leq 1)$ differentiates them.

4 Inference in $\mathcal{MLC}$

The complexity of the satisfiability problem for $\mathcal{MLC}$ and $\mathcal{PLC}$ have been studied in the literature. As we mention in Section 3, when dealing with complexity we should take care of whether numbers are coded in unary or binary. Let us call $\mathcal{L}^u$ and $\mathcal{L}^b$ the unary and binary coding, respectively. Then, the previously established results are as follows.

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[4] The proof goes through using the same models even if we add past operators to the language, as the bisimulation shown also satisfies the standard conditions Zig^{-1} and Zag^{-1} which preserve past operators [2].

Theorem 4. *1. $\mathcal{PLC}^u$-SAT is NP-complete [5].*
2. $\mathcal{MLC}^u$-SAT is ExpTime-complete [16, 8].
3. $\mathcal{PLC}^b$-SAT is NP-hard and in PSpace [4].
4. $\mathcal{MLC}^b$-SAT is ExpTime-hard and in 2-NExpTime [8].

Proof. Hardness in all cases is clear, we only comment on the upper bounds. The proof of 1) is via the polysize model property. The proof of 2) is by a linear satisfiability preserving translation into $\mathcal{H}(\mathsf{A})$ as we will show below. The proof of 3) is by a direct algorithm that solves satisfiability. The proof of 4) is by a linear satisfiability preserving translation into C^2, first order logic with only two variables and counting quantifiers.

We will not deal with the satisfiability problem in this paper, instead we will introduce the following inference task "exactly how many φ states are implied by the theory Γ?" Formally

Definition 4. *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas in $\mathcal{MLC}$, we define the function $|\varphi|$ in Γ as follows*

$$|\varphi| \textit{ in } \Gamma = \begin{cases} n & \textit{if } \Gamma \models \varphi = n \textit{ and } \Gamma \textit{ consistent} \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

We will show an algorithm that solves this task using any model building algorithm. In particular we will show how model building algorithms for $\mathcal{H}(\mathsf{A})$ like those proposed in [17, 18] can be used. We introduce first the notion of *negation normal form* for $\mathcal{MLC}$.

Definition 5. *Given $\varphi \in$ Forms the* negation normal form *of φ is obtained applying the following rules*

$$\begin{array}{r|l} \neg\neg\varphi \rightsquigarrow \varphi & \neg(\varphi \geq 0) \rightsquigarrow \bot \\ \neg(\varphi_1 \wedge \varphi_2) \rightsquigarrow (\neg\varphi_1) \vee (\neg\varphi_2) & \neg(\varphi \geq n) \rightsquigarrow \varphi \leq (n-1) \textit{ for } n > 0 \\ \neg(\varphi_1 \vee \varphi_2) \rightsquigarrow (\neg\varphi_1) \wedge (\neg\varphi_2) & \neg(\varphi \leq n) \rightsquigarrow \varphi \geq (n+1) \\ \neg\langle r\rangle\varphi \rightsquigarrow [r]\neg\varphi & \\ \neg[r]\varphi \rightsquigarrow \langle r\rangle\neg\varphi & \end{array}$$

As we mentioned in Section 3, every formula in $\mathcal{MLC}$ is equivalent to a formula where each counting operators has been *extracted* and it appears under the scope of neither modal nor counting operators. Each $\mathcal{MLC}$ formula is equivalent to its extracted, negation normal form. Let $\mathcal{MLC}^{en}$ be set of extracted formulas of $\mathcal{MLC}$ in negation normal form. We now present a translation from $\mathcal{MLC}^{en}$ to $\mathcal{H}(\mathsf{A})$ formulas. Tr_π works by traversing formulas and adding new nominals so that counting claims are preserved (π is used to ensure that we always introduce new nominals, initially π is set to the empty string; $i{:}\varphi$ is a satisfiability statement defined in $\mathcal{H}(\mathsf{A})$ as $\mathsf{A}(\neg i \vee \varphi)$).

$$\begin{array}{r|l} \mathsf{Tr}_\pi(p) = p & \mathsf{Tr}_\pi(\varphi \wedge \psi) = \mathsf{Tr}_{\pi 0}(\varphi) \wedge \mathsf{Tr}_{\pi 1}(\psi) \\ \mathsf{Tr}_\pi(\neg\varphi) = \neg\mathsf{Tr}_\pi(\varphi) & \mathsf{Tr}_\pi(\varphi \vee \psi) = \mathsf{Tr}_{\pi 0}(\varphi) \vee \mathsf{Tr}_{\pi 1}(\psi) \\ \mathsf{Tr}_\pi(\langle r\rangle\varphi) = \langle r\rangle\mathsf{Tr}_\pi(\varphi) & \mathsf{Tr}_\pi(\varphi \geq n) = (\bigwedge_{1 \leq i < j \leq n} x_i^\pi{:}\neg x_j) \wedge (\bigwedge_{1 \leq i \leq n} x_i^\pi{:}\varphi) \\ \mathsf{Tr}_\pi([r]\varphi) = [r]\mathsf{Tr}_\pi(\varphi) & \mathsf{Tr}_\pi(\varphi \leq n) = \mathsf{A}(\neg\varphi \vee \bigvee_{1 \leq i \leq n} x_i^\pi) \end{array}$$

in particular $Tr_\pi(\varphi \geq 0) = \top$ and $Tr_\pi(\varphi \leq 0) = \mathsf{A}(\neg\varphi)$.

Let us call $\varphi^{\mathcal{H}_\pi} = \mathsf{Tr}_\pi(\mathit{nnf}(\mathit{ext}(\varphi)))$ the formula obtained from the $\mathcal{MLC}$ formula φ by first extracting counting operators, transforming into negation normal form, and applying Tr_π; we write $\varphi^{\mathcal{H}}$ when π is the empty prefix.

Suppose now that $\mathcal{M}$ is a model satisfying $\varphi^{\mathcal{H}}$ returned by the model builder. We will show that counting has not been affected by the translation.

Definition 6. *We call a model $\mathcal{M}'$ a* naming extension *of $\mathcal{M}$ if it is a conservative extension of $\mathcal{M}$ for an extended language that only adds nominals.*

Theorem 5. *Let $\varphi \in \mathcal{MLC}$, and π an arbitrary prefix. Then $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w \models \varphi^{\mathcal{H}_\pi}$ for $\mathcal{M}'$ a naming extension of $\mathcal{M}$.*

Proof. We can disregard the extraction and negation normal form steps of the transformation since they are equivalence preserving.
[$\Rightarrow$] The atomic, negation and modal connectors cases are immediate. For any model $\mathcal{M}$ let us represent as $\mathcal{M}{+}N$ any naming extension of $\mathcal{M}$ where N is the function that assigns nominals to elements of the domain of $\mathcal{M}$. Assume $\mathcal{M}, w \models \varphi_1 \wedge \varphi_2$, i.e., $\mathcal{M}, w \models \varphi_1$ and $\mathcal{M}, w \models \varphi_2$. By induction hypothesis $\mathcal{M}{+}N_1, w \models \varphi_1^{\mathcal{H}_{\pi 0}}$ and $\mathcal{M}{+}N_2, w \models \varphi_2^{\mathcal{H}_{\pi 1}}$. As N_1 and N_2 are defined on different nominals we can obtain $N = N_1 \cup N_2$ and we have $\mathcal{M}{+}N, w \models \varphi_1^{\mathcal{H}_{\pi 0}} \wedge \varphi_2^{\mathcal{H}_{\pi 1}}$, and hence $\mathcal{M}{+}N, w \models (\varphi_1 \wedge \varphi_2)^{\mathcal{H}_\pi}$. The case for $\varphi_1 \vee \varphi_2$ is handled similarly.

Assume $\mathcal{M}, w \models \varphi \geq n$, i.e., there exist n different states v_1 to v_n such that for all $1 \leq i \leq n$, $\mathcal{M}, v_i \models \varphi$. For any π, choose $N = \bigcup_{1 \leq i \leq n}(x_i^\pi, v_i)$ to obtain $\mathcal{M}{+}N, w \models (\bigwedge_{1 \leq i < j \leq n} x_i^\pi{:}\neg x_j^\pi) \wedge (\bigwedge_{1 \leq i \leq n} x_i^\pi{:}\varphi)$ as needed.

Now, assume $\mathcal{M}, w \models \varphi \leq n$. Let v_1 to v_m $(m \leq n)$ be all the states of $\mathcal{M}$ satisfying φ. For any π, introduce n nominals x_1^π to x_n^π and a mapping N such that for $1 \leq i \leq n$ there exists j, $1 \leq j \leq m$ such that $(x_i^\pi, v_j) \in N$ (two nominals can be true in the same state). Then $\mathcal{M}{+}N, u \models \neg\varphi \vee \bigvee_{1 \leq i \leq n} x_i$ for u an arbitrary state, and $\mathcal{M}{+}N, w \models \varphi^{\mathcal{H}}$.

[$\Leftarrow$] Let $\varphi \in \mathcal{MLC}$ and π an arbitrary prefix, and $\mathcal{M}'$ a naming extension of $\mathcal{M}$ such that $\mathcal{M}', w \models \varphi^{\mathcal{H}_\pi}$. If φ is a modal formula the implication is trivial.

Assume $\mathcal{M}', w \models (\varphi \geq n)^{\mathcal{H}_\pi}$. By definition $\mathcal{M}', w \models (\bigwedge_{1 \leq i < j \leq n} x_i^\pi{:}\neg x_j^\pi) \wedge (\bigwedge_{1 \leq i \leq n} x_i^\pi{:}\varphi)$. Since x_1^π to x_n^π are all true at different states $\mathcal{M}, w \models \varphi \geq n$.

Assume $\mathcal{M}', w \models (\varphi \leq n)^{\mathcal{H}(\pi)}$, i.e., $\mathcal{M}', w \models \mathsf{A}(\neg\varphi \vee \bigvee_{1 \leq i \leq n} x_i^\pi)$. Then an arbitrary u of $\mathcal{M}'$, $\mathcal{M}', u \models \neg\varphi \vee \bigvee_{1 \leq i < n} x_i^\pi$. Hence, either $\mathcal{M}', u \models \neg\varphi$ or $\mathcal{M}', u \models x_i^\pi$ for a given $i \in [[1..m]]$, ie $\{u\} = V(x_i^\pi)$ for $i \in [[1..m]]$. So there can not be more than n distinct states satisfying φ in $\mathcal{M}'$ and $\mathcal{M}, w \models \varphi \leq n$. $\square$

Thus we can say that for a given $\mathcal{MLC}$ formula φ, a model of $\varphi^{\mathcal{H}}$ is a model of φ. We can now present the algorithm that carries out the reasoning task of counting. Given P a decision procedure for $\mathcal{H}(\mathsf{A})$, $\mathit{\Gamma}$ a finite set of $\mathcal{MLC}$ formulas and φ a $\mathcal{MLC}$ formula:

```
if $P(\Gamma^{\mathcal{H}})$ returns UNSAT then
  return 'undefined'
else
  let n = $|\|\varphi\|^{\mathcal{M}}|$ for $\mathcal{M}$ a model returned by $P$
  if $P((\Gamma \wedge \neg(\varphi = n))^{\mathcal{H}})$ returns UNSAT then
    return n
  else
    return 'undefined'
  end if
end if
```

Intuitively, our counting algorithm uses a model of the theory Γ to have a candidate answer n to the question "how many φ are implied by Γ?". We then test satisfiability of $(\Gamma \wedge \neg(\varphi = n))^{\mathcal{H}}$ to get the answer.

Theorem 6. *The algorithm above computes* $|\varphi|$ *in* Γ.

5 Conclusions

In this paper we investigated various aspects of modal logics containing the counting quantifiers $(\varphi \geq n)$ and $(\varphi \leq n)$, motivated by the natural language application of representing and querying plural objects in a discourse.

These quantifiers have been introduced before in different areas (generalized quantifiers, modal logics, and description logics), and some of their previously known properties have been outlined (existence of extracted normal forms, complexity of the satisfiability problem, etc.). In this article we investigate expressive power and inference.

With respect to the former, we introduce the notion of $\mathcal{MLC}$ bisimulations, prove that it preserves $\mathcal{MLC}$ formulas and that it characterizes $\mathcal{MLC}$-equivalent finite models. A natural next step would be to investigate "van Benthem characterization" results [19]. I.e., to verify whether any formula of the first-order language with equality (in the appropriate signature) invariant under $\mathcal{MLC}$ bisimulations is equivalent to the translation of an $\mathcal{MLC}$ formula. We strongly conjecture that this is the case.

With respect to inference, we defined a new task that given a theory Γ and a formula φ returns the cardinality of the extension of φ in any model of Γ if such cardinality is fixed to be a finite natural number. We show that this task can be solved in terms of a calculus for the hybrid logic $\mathcal{H}(\mathsf{A})$ that can return a model for any satisfiable formula (e.g., tableaux based calculi as those defined by [17, 18]). The proposed algorithm involves a translation into $\mathcal{H}(\mathsf{A})$ that might return an exponentially larger formula even when numbers are coded in unary. We conjecture that the polynomial satisfiability preserving translation of [14] could be used instead (but assuming, again, that numbers are coded in unary). The complexity of the problem when numbers are coded in binary is open. As we mentioned in Section 4, the complexity of satisfiability for $\mathcal{MLC}$ and $\mathcal{PLC}$ when numbers are coded in unary has been established [5, 16, 8]. On the other hand, to our knowledge the problem is still open when numbers are given in binary.

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