Hybrid Logic is the Bounded Fragment of First Order Logic

CARLOS ARECES, *University of Amsterdam, The Netherlands, carlos@wins.uva.nl*

PATRICK BLACKBURN, *Universität des Saarlandes, Germany, patrick@coli.uni-sb.de*

MAARTEN MARX, *University of Amsterdam, The Netherlands, marx@wins.uva.nl*

Abstract

Hybrid languages are extended modal languages which can refer to (or even quantify over) worlds. The use of strong hybrid languages dates back to at least [Pri67], but recent work (for example [BS98, BT99]) has focussed on a more constrained system called $\mathcal{H}(\downarrow, @)$. The purpose of the present paper is to show in detail that $\mathcal{H}(\downarrow, @)$ is a modally natural system. We study its expressivity, and provide both model theoretic characterizations (via a restricted notion of Ehrenfeucht-Fraïssé game, and an enriched notion of bisimulation) and a syntactic characterization (in terms of bounded formulas). The key result is that $\mathcal{H}(\downarrow, @)$ corresponds precisely to the first-order fragment which is invariant for generated submodels.

Keywords: Modal and tense logic, bisimulations, first order logic.

1 Introduction

In their simplest form, hybrid languages are modal languages which use *formulas* to refer to worlds. To build a simple hybrid language, take an ordinary language of propositional modal logic (built over some collection of propositional variables p, q, r, and so on) and add a second type of atomic formula. These new atoms are called *nominals*, and are typically written i, j and k. Both types of atom can be freely combined to form more complex formulas in the usual way; for example,

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

is a well formed formula. And now for the key idea: insist that *each nominal must be true at exactly one world in any model.* Thus a nominal names a world by being true there and nowhere else. This simple idea gives rise to richer logics (note, for example, that the previous formula is valid: if the antecedent is satisfied at a world m, then the unique world named by i must be accessible from m, and both p and q must be true there) and enables us to define classes of frames that ordinary modal languages cannot (we'll see some examples later).

Once the idea of using "formulas as terms" has been noted (Arthur Prior [Pri67], influenced by unpublished work of C. A. Meredith, seems to have been the first to grasp its potential) the way lies open for further enrichments. The most obvious is to regard nominals not as names but as *variables over individual worlds*, and to add quantifiers. That is, we now allow expressions like

$$\forall x.\Diamond(x \wedge \exists y.\Diamond(y \wedge \Diamond y))$$

to be formed. This sentence is satisfied at a world m if and only if from every world x that is accessible from m we can reach at least one reflexive world y. No formula with this property exists in ordinary modal languages, or even in modal languages enriched with nominals. Unsurprisingly, if we are allowed to quantify over worlds in this manner, it is straightforward to define hybrid languages that offer first-order expressivity over models. Early work on hybrid languages (notably that of Bull [Bul70] and Passy and Tinchev [PT85, PT91]) was largely concerned with such systems.

The idea of binding variables to worlds underlies much current work on hybrid languages, but for many purposes the $\forall$ binder is arguably too strong: languages which employ $\forall$ obscure the *locality* intuition central to Kripke semantics. Fundamental to Kripke semantics is the relativization of semantic evaluation to worlds. That is, to evaluate a modal formula we need to specify some world m (the *current* world) and begin evaluation *there*. The function of the modalities is to scan the worlds accessible from m, the worlds accessible from those worlds, and so on; in short, m is the starting point for step-wise local exploration of the model. Languages which allow variables to be bound to *arbitrary* worlds (some of which may not be reachable from m by chaining through the accessibility relation) don't mesh well with this intuition.

For this reason, recent work on hybrid languages has focussed on a local language called $\mathcal{H}(\downarrow, @)$. This extends the simplest type of hybrid language (propositional modal logic plus nominals) with two new mechanisms, $\downarrow$ and @. Now $\downarrow$ does bind variables to worlds, but (unlike $\forall$) it does so in an intrinsically *local* way:

> *The* $\downarrow$ *binder binds variables to the* current *world. In essence it enables us to create a name for the here-and-now.*

The @ operator (which does *not* bind variables) is a natural counterpart to $\downarrow$. Whereas $\downarrow$ "stores" the current world (by binding a variable to it), @ enables us to "retrieve" worlds. More precisely, a formula of the form $@_x\varphi$ is an instruction to move to the world labeled by the variable x and evaluate φ there. Previous work on $\mathcal{H}(\downarrow, @)$ has concentrated on relating it to other hybrid languages [BS98], studying it axiomatically [BT99], and developing analytic proof techniques [Bla98, Tza98]. Taken together, this work suggests $\mathcal{H}(\downarrow, @)$ and certain of its sublanguages (notably $\mathcal{H}(@)$) are important systems. The purpose of the present paper is to demonstrate in detail that this impression is justified.

We do so as follows. After defining $\mathcal{H}(\downarrow, @)$ and noting some basic results in Section 2, we turn in Section 3 to the task of characterizing its expressivity. The key result to emerge is this: $\mathcal{H}(\downarrow, @)$ is not merely local, it is the language which *characterizes* locality. More precisely, $\mathcal{H}(\downarrow, @)$ corresponds to the fragment of first-order logic which is invariant under generated submodels. Previous discussions of $\mathcal{H}(\downarrow, @)$ have stressed that it is "modally natural"; our characterization confirms this impression and makes it precise. In Section 4 we discuss the consequences of this characterization for frame-definability, completeness, and tense logic.

The paper is largely self contained, but as the existing literature on hybrid languages is small and scattered it seems appropriate to give the reader a swift overview of what is available. First, two early papers on ∀-based hybrid languages (namely [Bul70] and [PT91]) deserve to be far more widely read: both contain important technical ideas and interesting motivation for the use of hybrid languages. Second, some work has been done on very basic hybrid languages (that is, modal or tense languages enriched with nominals, but with no additional mechanisms such as ↓, @, or ∀); the main references here are [GG93] and [Bla93]. Third, while [BS98] and [BT99] are the basic references for $\mathcal{H}(\downarrow, @)$, an interesting discussion of ↓ as part of a stronger system can be found in [Gor96]. Finally, in addition to the proof theoretical investigations of [Bla98] and [Tza98], there is also [Sel91, Sel97]. Seligman's work deals with stronger (∀-based) systems, but many of the key ideas underlying hybrid deduction (in particular, the deductive significance of @) were first explored here, and they are still the only source we know of for discussion of natural deduction techniques for hybrid logics.

2 Preliminaries

In this section we define the syntax and semantics of $\mathcal{H}(\downarrow, @)$ and note some of its basic properties.

Definition 2.1 (Language)
Let PROP $= \{p_1, p_2, \ldots\}$ be a countable set of *propositional variables*, NOM $= \{i_1, i_2, \ldots\}$ a countable set of *nominals*, and WVAR $= \{x_1, x_2, \ldots\}$ a countable set of *world variables*. We assume that PROP, NOM and WVAR are pairwise disjoint. We call WSYM = NOM ∪ WVAR the set of *world symbols*, and ATOM = PROP ∪ NOM ∪ WVAR the set of *atoms*. The *well-formed formulas* of the hybrid language (over the signature ⟨PROP, NOM, WVAR ⟩) are

$$\varphi := \top \mid \bot \mid a \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \Box\varphi \mid \ \downarrow x_j.\varphi \mid @_s\varphi$$

where $a \in$ ATOM, $x_j \in$ WVAR and $s \in$ WSYM. Let $\mathcal{L}$ be the set of all well-formed formulas. For $\varphi \in \mathcal{L}$, PROP(φ), NOM(φ) and WVAR(φ) denote, respectively, the set of propositional variables, nominals, and world variables which occur in φ. $I\!\!P(\varphi)$ will denote PROP(φ) ∪ NOM(φ), and will be called the *language* of φ.

In what follows we assume that a signature ⟨PROP, NOM, WVAR⟩, and hence $\mathcal{L}$, has been fixed. We usually write p, q and r for propositional variables, i, j and k for nominals, and x, y and z for world variables. As usual, $\Diamond\varphi$ is defined to be $\neg\Box\neg\varphi$

Note that all three types of atomic symbol are *formulas*. Further, note that the above syntax is simply that of ordinary unimodal propositional modal logic extended by the clauses for $\downarrow x_j.\varphi$ and $@_s\varphi$. Finally, the difference between nominals and world variables is simply this: nominals cannot be bound by ↓, whereas world variables can. In fact, nominals could be dispensed with (it is always possible to make do with free world variables instead) but for some purposes it is useful to have a special kind of world symbol that can't be accidentally bound.

Definition 2.2
The notions of *free* and *bound* world variable are defined in the manner familiar from first-order logic, with ↓ as the only binding operator. Similarly, other syntactic notions (such as *substitution*, and of a world symbol t *being substitutable for* x *in* φ) are defined as the corresponding notions in first-order logic. We use $\varphi[t/s]$ to denote

the formula obtained by replacing all free instances of the world symbol t by the world symbol s.

A *sentence* is a formula containing no free world variables. Furthermore, a formula is *pure* if it contains no propositional variables, and *nominal-free* if it contains no nominals.

DEFINITION 2.3 (Semantics)
A (hybrid) *model* $\mathfrak{M}$ for $\mathcal{L}$ is a triple $\mathfrak{M} = \langle M, R, V\rangle$ such that M is a non-empty set, R a binary relation on M, and V : PROP $\cup$ NOM $\rightarrow$ Pow(M) such that for all nominals $i \in$ NOM, $V(i)$ is a singleton subset of M. (We use calligraphic letters $\mathfrak{M}$ for models, italic roman M for their domains. When necessary, we use sub or supra-indexes.) We usually call the elements of M worlds (though sometimes we call them times or states), and we call R the *accessibility relation* and V the *valuation*. A *frame* is a pair $\mathfrak{M} = \langle M, R\rangle$; that is, a frame is a model without a valuation.

An *assignment* g for $\mathfrak{M}$ is a mapping g : WVAR $\rightarrow M$. Given an assignment g, we define the assignment g^i_m by $g^i_m(x_j) = g(x_j)$ for $j \neq i$ and $g^i_m(x_i) = m$. We say that g^i_m is *an x_i-variant of g*.

Let $\mathfrak{M} = \langle M, R, V\rangle$ be a model, $m \in M$, and g an assignment. For any atom a, let $[V, g](a) = \{g(a)\}$ if a is a world variable, and $V(a)$ otherwise. Then the *forcing relation* is defined as follows

$\mathfrak{M}, g, m \Vdash \top$		
$\mathfrak{M}, g, m \not\Vdash \bot$		
$\mathfrak{M}, g, m \Vdash a$	iff	$m \in [V, g](a)$, $a \in$ ATOM
$\mathfrak{M}, g, m \Vdash \neg\varphi$	iff	$\mathfrak{M}, g, m \not\Vdash \varphi$
$\mathfrak{M}, g, m \Vdash \varphi \wedge \psi$	iff	$\mathfrak{M}, g, m \Vdash \varphi$ and $\mathfrak{M}, g, m \Vdash \psi$
$\mathfrak{M}, g, m \Vdash \Box\varphi$	iff	$\forall m'(Rmm' \Rightarrow \mathfrak{M}, g, m' \Vdash \varphi)$
$\mathfrak{M}, g, m \Vdash \downarrow x_i.\varphi$	iff	$\mathfrak{M}, g^i_m, m \Vdash \varphi$
$\mathfrak{M}, g, m \Vdash @_s\varphi$	iff	$\mathfrak{M}, g, m' \Vdash \varphi$, where $[V, g](s) = \{m'\}$, $s \in$ WSYM.

When $\mathfrak{M}$ and g are understood by context we will simply write $m \Vdash \varphi$ for $\mathfrak{M}, g, m \Vdash \varphi$. We write $\mathfrak{M}, g \Vdash \varphi$ iff for all $m \in M$, $\mathfrak{M}, g, m \Vdash \varphi$. We write $\mathfrak{M} \models \varphi$ iff for all g, $\mathfrak{M}, g \Vdash \varphi$.

The first six clauses are essentially the standard Kripke forcing relation for propositional modal logic; in fact the only difference is that whereas the standard definition relativizes semantic evaluation to worlds m, we relativize to variable assignments g as well. Note that the clause for atoms covers all three types of symbol (propositional variables, nominals, and world variables) and that given any model $\mathfrak{M}$ and assignment g, any world symbol (whether it is a nominal or a world variable) will be forced at a *unique* world; this is an immediate consequence of the way we defined valuations and assignments. As promised in the introduction, $\downarrow$ binds world variables to the world where evaluation is being performed (the *current world*), and $@_s$ shifts evaluation to the world named by s. Just as in first-order logic, if φ is a *sentence* it is irrelevant which assignment g is used to perform evaluation: $\mathfrak{M}, g, m \Vdash \varphi$ for *some* assignment g iff $\mathfrak{M}, g, m \Vdash \varphi$ for *all* assignments g. Hence for sentences the relativization to assignments of the forcing relation can be dropped, and we simply write $\mathfrak{M}, m \Vdash \varphi$ instead of $(\forall g).\mathfrak{M}, g, m \Vdash \varphi$.

A formula φ is *satisfiable* if there is a model $\mathfrak{M}$, an assignment g on $\mathfrak{M}$, and a world $m \in M$ such that $\mathfrak{M}, g, m \Vdash \varphi$. A formula φ is *valid* if for all models $\mathfrak{M}$, $\mathfrak{M} \models \varphi$.

A formula φ is a *local consequence* of a set of formulas T if for some finite subset $\{\varphi_1, \ldots, \varphi_n\}$ of T, $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \varphi$ is valid. A formula φ is a *global consequence* of a set of formulas T if for all models $\mathfrak{M}$, $\mathfrak{M} \models \varphi$ only if for all $\psi \in T$, $\mathfrak{M} \models \psi$. We denote local consequence by $T \models \varphi$ and global consequence by $T \models^{glo} \varphi$. As in ordinary propositional modal local, local consequence is strictly stronger than global consequence.

$\mathcal{H}(\downarrow, @)$ offers us considerable expressive power over models. For example we can define the *Until* operator:

$$\mathit{Until}(\varphi, \psi) \ := \ \downarrow x.\Diamond \downarrow y.@_x(\Diamond(y \wedge \varphi) \wedge \Box(\Diamond y \rightarrow \psi)).$$

Note how this works: we name the current world x, use $\Diamond$ to move to an accessible world, which we name y, and then use @ to jump us back to x. We then use the modalities to insist that (1) φ holds at the world named y, and (2) ψ holds at all successors of the current world that precede this y-labeled world.

But there is an obvious (and modally natural) limit to the expressive power $\mathcal{H}(\downarrow, @)$ gives us: any nominal-free sentence is preserved under the formation of *point-generated* (or *rooted*) submodels. That is, if a sentence φ is satisfied at a world m in a model $\mathfrak{M}$, and we form a submodel $\mathfrak{M}_m$ by discarding from $\mathfrak{M}$ all the worlds that are *not* reachable by making a finite (possibly empty) sequence of transitions from m, then $\mathfrak{M}_m$ also satisfies φ at m. (The key point to observe is that in any subformula of φ of the form $@_t\psi$, t must be a world variable bound by some previous occurrence of $\downarrow$. As $\downarrow$ binds to the current world, t is bound to some world in the submodel generated by m, thus φ is unaffected by the restriction to $\mathfrak{M}_m$.) That is, $\mathcal{H}(\downarrow, @)$ is genuinely local: only reachable worlds are relevant to semantic evaluation. In the following section we shall generalize this observation (we have not merely preservation, but *invariance*) and show that it characterizes the expressivity of $\mathcal{H}(\downarrow, @)$.

$\mathcal{H}(\downarrow, @)$ also offers us considerable expressive power with respect to *frames*. Modal logicians like to view modal languages as tools for talking about frames, and they do so via the concept of frame validity:

Definition 2.4
A formula φ is *valid on a frame* $\mathfrak{F} = \langle M, R\rangle$ if for every valuation V on $\mathfrak{F}$, and every assignment g on $\mathfrak{F}$, and every $m \in M$, $\langle \mathfrak{F}, V\rangle, g, m \Vdash \varphi$. A formula is *valid on a class of frames* F if it is valid on every frame $\mathfrak{F}$ in F. A formula φ *defines a class of frames* if it is valid on precisely the frames in F, and it defines a *property* of frames (for example, transitivity) if it defines the class of frames with that property.

Many interesting properties are definable using pure, nominal-free, sentences:

$\downarrow x.\Box\neg x$	*Irreflexivity*
$\downarrow x.\Box\Box\neg x$	*Asymmetry*
$\downarrow x.\Box(\Diamond x \rightarrow x)$	*Antisymmetry*
$\downarrow x.\Box \downarrow y.@_x\Diamond\Diamond y$	*Density*
$\downarrow x.\Box\Box \downarrow y.@_x\Diamond y$	*Transitivity*
$\downarrow x.\Diamond \downarrow y.@_x(\Box\Box\neg y \wedge \Box \downarrow z.@_y(z \vee \Diamond z))$	*Right-Discreteness*

With the exception of transitivity and density, none of these properties are definable in ordinary modal logic. In Section 4 we shall characterize the classes of frames that pure, nominal-free, sentences can define.

[BT99] provides the following complete axiom system for $\mathcal{H}(\downarrow, @)$:

DEFINITION 2.5 (Axiomatization)
Let φ, ψ be formulas, v a metavariable over world variables and s, t metavariables over world symbols. The hybrid logic $\mathsf{K}[\mathcal{H}(\downarrow, @)]$ is the smallest subset of $\mathcal{L}$ containing all instances of propositional tautologies, all instances of the following axiom schemes and closed under the following deduction rules

K. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
N. $\vdash \varphi \Rightarrow \vdash\Box\varphi$.
MP. $\vdash \varphi \to \psi, \vdash \varphi \Rightarrow \vdash\psi$.

Q1. $\downarrow v.(\varphi \to \psi) \to (\varphi \to \downarrow v.\psi)$, φ without free occurrences of v.
Q2. $\downarrow v.\varphi \to (s \to \varphi[v/s])$, s substitutable for v in φ.
Q3. $\downarrow v.(v \to \varphi) \to \downarrow v.\varphi$.
Self Dual$_\downarrow$. $\downarrow v.\varphi \leftrightarrow \neg \downarrow v.\neg\varphi$.
N$_\downarrow$. $\vdash \varphi \Rightarrow \vdash \downarrow v.\varphi$.

K$_@$. $@_s(\varphi \to \psi) \to (@_s\varphi \to @_s\psi)$.
Self Dual$_@$. $@_s\varphi \leftrightarrow \neg @_s\neg\varphi$.
Introduction. $(s \wedge \varphi) \to @_s\varphi$.
Label. $@_s s$.
Nom. $@_s t \to (@_t\varphi \to @_s\varphi)$.
Swap. $@_s t \leftrightarrow @_t s$.
Scope. $@_t @_s\varphi \leftrightarrow @_s\varphi$.
N$_@$. $\vdash \varphi \Rightarrow \vdash @_s\varphi$.

Back. $\Diamond @_s\varphi \to @_s\varphi$.
Bridge. $(\Diamond s \wedge @_s\varphi) \to \Diamond\varphi$.

Paste-0. $\vdash @_s(t \wedge \varphi) \to \psi \Rightarrow \vdash @_s\varphi \to \psi$, $t \in \mathrm{WSYM}\backslash\mathrm{WSYM}(\varphi, \psi, s)$.
Paste-1. $\vdash @_s\Diamond(t \wedge \varphi) \to \psi \Rightarrow \vdash @_s\Diamond\varphi \to \psi$, $t \in \mathrm{WSYM}\backslash\mathrm{WSYM}(\varphi, \psi, s)$.

The completeness result proved in [BT99] is very general: not only does this axiomatization generate all valid formulas, but it automatically extends to many stronger logics. In particular, if we add a pure, nominal-free, sentence φ as an additional axiom, the resulting system is strongly complete with respect to the class of frames that φ defines. In Section 4 we shall characterize the completeness results covered by such extensions.

3 Characterizing $\mathcal{H}(\downarrow, @)$

In this section we characterize (the first-order language corresponding to) $\mathcal{H}(\downarrow, @)$. We begin by providing a *syntactic* characterization. In particular, we shall first extend the *standard translation* ST of modal logic into first-order logic (cf. [Ben83]) to $\mathcal{H}(\downarrow, @)$. It will be clear that the range of our translation lies in a certain *bounded fragment*, and we shall define a reverse translation HT which maps the bounded fragment back into the the hybrid language. Thus we are free to think either in terms of $\mathcal{H}(\downarrow, @)$ or the corresponding bounded fragment.

But how are these languages characterized *semantically*? It should be clear that $\mathcal{H}(\downarrow, @)$ is a genuine hybrid of modal and first-order ideas (after all, it combines Kripke semantics with the idea of binding variables to worlds) thus there are two obvious ways to proceed. The first is essentially first-order: we could look for a weaker notion of Ehrenfeucht-Fraïssé game. The second is essentially modal: we could try looking for a stronger notion of bisimulation. We shall pursue *both* options. As we shall see, both yield natural notions of equivalence between models, and by relating them (and drawing on our syntactic characterization) we can provide an detailed picture of what $\mathcal{H}(\downarrow, @)$ offers.

3.1 Translations

We focus on two kinds of signature for first-order logic with equality. First we have *modal* signatures (familiar from modal correspondence theory [Ben83]) which consist of one binary predicate R, countably many unary predicates, and no constant symbols. It will be convenient to make the set of first-order variables at our disposal explicit in the signature (just as we did when we defined hybrid signatures in Definition 2.1) thus, a modal signature has the form $\langle \{R\} \cup \text{UN-REL}, \{\}, \text{VAR}\rangle$. A *hybrid* signature is an expansion of the modal signature with countably many constant symbols, thus hybrid signatures have the form $\langle \{R\} \cup \text{UN-REL}, \text{CONS}, \text{VAR}\rangle$. Note that any hybrid model $\mathfrak{M} = \langle M, R, V\rangle$ can be regarded as a first-order model with domain M, for the accessibility relation R can be used to interpret the binary predicate R, unary predicates can be interpreted by the subsets V assigns to propositional variables, and constants (if any) can be interpreted by the worlds that nominals name. So we let the context determine whether we are thinking of first-order or hybrid models, and continue to use the notation $\mathfrak{M} = \langle M, R, V\rangle$.

We first extend the well-known *standard translation* to $\mathcal{H}(\downarrow, @)$. The translation ST from the hybrid language over $\langle \text{PROP}, \text{NOM}, \text{WVAR}\rangle$ into first-order logic over the signature $\langle \{R\} \cup \{P_j \mid p_j \in \text{PROP}\}, \text{NOM}, \text{WVAR} \cup \{x, y\}\rangle$ is defined by mutual recursion between two functions ST_x and ST_y. Recall that $\varphi[x/y]$ means "replace all free instances of x by y".

$ST_x(p_j)$	$=$	$P_j(x)$, $p_j \in \text{PROP}$.	$ST_y(p_j)$	$=$	$P_j(y)$, $p_j \in \text{PROP}$.
$ST_x(i_j)$	$=$	$x = i_j$, $i_j \in \text{NOM}$.	$ST_y(i_j)$	$=$	$y = i_j$, $i_j \in \text{NOM}$.
$ST_x(x_j)$	$=$	$x = x_j$, $x_j \in \text{WVAR}$.	$ST_y(x_j)$	$=$	$y = x_j$, $x_j \in \text{WVAR}$.
$ST_x(\neg\varphi)$	$=$	$\neg ST_x(\varphi)$.	$ST_y(\neg\varphi)$	$=$	$\neg ST_y(\varphi)$.
$ST_x(\varphi \wedge \psi)$	$=$	$ST_x(\varphi) \wedge ST_x(\psi)$.	$ST_y(\varphi \wedge \psi)$	$=$	$ST_y(\varphi) \wedge ST_y(\psi)$.
$ST_x(\Diamond\varphi)$	$=$	$\exists y.(Rxy \wedge ST_y(\varphi))$.	$ST_y(\Diamond\varphi)$	$=$	$\exists x.(Ryx \wedge ST_x(\varphi))$.
$ST_x(\downarrow x_j.\varphi)$	$=$	$(ST_x(\varphi))[x_j/x]$.	$ST_y(\downarrow x_j.\varphi)$	$=$	$(ST_y(\varphi))[x_j/y]$.
$ST_x(@_s\varphi)$	$=$	$(ST_x(\varphi))[x/s]$.	$ST_y(@_s\varphi)$	$=$	$(ST_y(\varphi))[y/s]$.

For m an element in the domain of a given model $\mathfrak{M}$ we will often write $ST_m(\varphi)$ as shorthand for $ST_x(\varphi)[m]$. This translation differs from the one given in [BS98]; these authors handle $\downarrow$ and @ as follows:

$$\begin{array}{lcl} ST_x(\downarrow x_j.\varphi) & = & \exists x_j.(x = x_j \wedge ST_x(\varphi)). \\ ST_x(@_s\varphi) & = & ST_s(\varphi),\ s \in \text{NOM} \cup \text{VAR}. \end{array}$$

Now the [BS98] translation makes the quantificational effect of $\downarrow$ clear, but our translation draws attention to another perspective: in adding $\downarrow$ and @ we have in effect enriched the modal language with an *explicit substitution operator*. Such operators are used in the study of cylindric algebras, and were added to cylindric modal logic in [Ven94].

The link between $\downarrow$ and explicit substitution can be made even more clear if we expand the first-order language with an explicit substitution operator (like s^i_j in the theory of cylindric algebras) and adjust our definition of ST to take advantage of it. We do this as follows. Add the following clause to the grammar generating the first-order language:

if φ is a formula and x, y are variables, then $S^x_y\varphi$ is a formula.

Interpret S^x_y as follows:

$$\mathfrak{M} \models S^x_y\varphi[g] \Leftrightarrow \begin{cases} \mathfrak{M} \models \varphi[g] & \text{for } x = y \\ \mathfrak{M} \models \varphi[g^x_y] & \text{for } x \neq y. \end{cases}$$

Clearly $S^x_y\varphi$ and $\varphi[x/y]$. This enrichment of first-order logic can be axiomatized by adding the following axiom schemata to a complete axiomatization of first-order logic with equality:

$$\begin{array}{lcl} S^x_x\varphi & \leftrightarrow & \varphi \\ S^x_y\varphi & \leftrightarrow & \exists x.(x = y \wedge \varphi) \quad \text{for } x \neq y. \end{array}$$

And now we can give transparent translations of $\downarrow$ and @:

$$\begin{array}{lcl} ST_x(\downarrow x_j.\varphi) & = & S^{x_j}_x ST_x(\varphi) \\ ST_x(@_s\varphi) & = & S^x_s ST_x(\varphi). \end{array}$$

Note that theorems like $\downarrow v.@_v\varphi \leftrightarrow \downarrow v.\varphi$ follow more or less immediately, for $ST_x(\downarrow v.@_v\varphi) = S^v_x S^x_v ST_x(\varphi)$, but $S^v_x S^x_v\varphi \equiv S^v_x S^x_x\varphi = S^v_x\varphi$, which is equivalent to $S^v_x ST_x(\varphi)$. However we shall stick with our original formulation of ST in what follows.

Proposition 3.1 (ST Preserves Truth)
Let φ be a hybrid formula, then for all hybrid models $\mathfrak{M}$, $m \in M$ and assignments g, $\mathfrak{M}, g, m \Vdash \varphi$ iff $\mathfrak{M} \models ST_m(\varphi)[g]$.

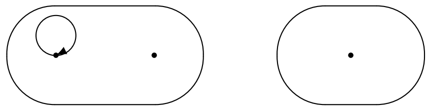
Proof. A straightforward extension of the induction familiar from ordinary modal logic. The only cases which are new are $ST_x(\downarrow x_j.\varphi)$ and $ST_x(@_s\varphi)$. But $\mathfrak{M}, g, m \Vdash \downarrow x_j.\varphi$, iff $\mathfrak{M}, g^{x_j}_m, m \Vdash \varphi$, by IH iff, $\mathfrak{M} \models ST_m(\varphi)[g^{x_j}_m]$, iff $\mathfrak{M} \models (ST_m(\varphi))[x_j/x]\ \ [g]$. The argument for $ST_x(@_s\varphi)$ is similar. ∎

Now for the interesting question: what is the *range* of ST? In fact it belongs to a *bounded* fragment of first-order logic. Given a first-order signature $\langle\{R\} \cup \text{UN-REL}, \text{CONS}, \text{VAR}\rangle$ we define the *bounded* fragment as the set of formulas generated by the following grammar:

$$\varphi = Rtt' \mid P_jt \mid t = t' \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \exists x_i.(Rtx_i \wedge \varphi) \text{ (for } x_i \neq t).$$

where $x_i \in \text{VAR}$ and $t, t' \in \text{VAR} \cup \text{CONS}$.

The side-condition on the generation of existentially quantified formulas is crucial: it prevents sentences like $\exists x.(Rxx \wedge x = x)$ from falling into the fragment. The sentence $\exists x.Rxx$ is probably the simplest example of a first-order sentence which is *not invariant for generated submodels* (or subframes). In fact it is not even *preserved* under the formation of generated submodels, for it is true in the model $\mathfrak{M}_1$ but not in the generated submodel $\mathfrak{M}_2$:



$\mathfrak{M}_1$ $\mathfrak{M}_2$

Clearly ST generates formulas in the bounded fragment. Crucially, however, we can also translate the bounded fragment into $\mathcal{H}(\downarrow, @)$. The translation HT from the bounded fragment over $\langle \{R\} \cup \text{UN-REL, CONS, VAR} \rangle$ into the hybrid language over $\langle \text{UN-REL}, \text{CONS}, \text{VAR} \rangle$ is defined as follows:

$$
\begin{array}{lcl}
HT(Rtt') & = & @_t \Diamond t'. \\
HT(P_j t) & = & @_t p_j. \\
HT(t = t') & = & @_t t'. \\
HT(\neg\varphi) & = & \neg HT(\varphi). \\
HT(\varphi \wedge \psi) & = & HT(\varphi) \wedge HT(\psi). \\
HT(\exists v.(Rtv \wedge \varphi)) & = & @_t \Diamond \downarrow v.HT(\varphi).
\end{array}
$$
(Here $t, t' \in \text{VAR} \cup \text{CONS}$.)

By construction, $HT(\varphi)$ is a hybrid formula, but furthermore it is a boolean combination of @-formulas (formulas whose main operator is @). So, because @ commutes with all the booleans (recall the Self Dual$_@$ axiom), it is equivalent to a formula whose main operator is @. We can now prove the following strong truth preservation result. Recall that $\mathfrak{M}, g \Vdash \phi$, abbreviates "(for all $m \in M$) $\mathfrak{M}, g, m \Vdash \phi$".

Proposition 3.2 (HT Preserves Truth)
Let φ be a bounded formula. Then for every first-order model $\mathfrak{M}$ and for every assignment g, $\mathfrak{M} \models \varphi[g]$ iff $\mathfrak{M}, g \Vdash HT(\varphi)$.

Proof. The proof uses the following fact about formulas whose main operator is @:

$$\text{(there exists an } m), \mathfrak{M}, g, m \Vdash @_s\varphi \text{ if and only if } \mathfrak{M}, g \Vdash @_s\varphi.$$

Again there is only one interesting case: $HT(\exists v.(Rtv \wedge \varphi))$. Now $\mathfrak{M} \models \exists v.(Rtv \wedge \varphi)[g]$ iff $\mathfrak{M} \models (Rtv \wedge \varphi)[g^v_{\mathbf{m}}]$ for some $\mathbf{m} \in M$. Let $\mathbf{t}$ be the interpretation of t in $\mathfrak{M}$ under $g^v_{\mathbf{m}}$. Because of the restriction on variables in bounded quantification, $t \neq v$, whence $\mathbf{t}$ is also the interpretation of t in $\mathfrak{M}$ under g. So $R\mathbf{tm}$ holds in $\mathfrak{M}$ and $\mathfrak{M} \models \varphi[g^v_{\mathbf{m}}]$. By the inductive hypothesis, $\mathfrak{M}, g^v_{\mathbf{m}} \Vdash HT(\varphi)$ iff $\mathfrak{M}, g, \mathbf{m} \Vdash \downarrow v.HT(\varphi)$, iff $\mathfrak{M}, g, \mathbf{t} \Vdash \Diamond \downarrow v.HT(\varphi)$ iff $\mathfrak{M}, g, \mathbf{t} \Vdash @_t\Diamond \downarrow v.HT(\varphi)$ iff $\mathfrak{M}, g \Vdash @_t\Diamond \downarrow v.HT(\varphi)$. ■

As simple corollaries we have:

Corollary 3.3
Let $\varphi(x)$ be a bounded formula with only x free, then for all models $\mathfrak{M}$ and for all $m \in M$, $\mathfrak{M} \models \varphi(m)$ iff $\mathfrak{M}, m \Vdash \downarrow x.HT(\varphi)$.

Corollary 3.4
Let φ be a first-order formula in the hybrid signature. Then the following are equivalent.

i. φ is equivalent to the standard translation of a hybrid formula.
ii. φ is equivalent to a formula in the bounded fragment.

Moreover, there are effective translations between $\mathcal{H}(\downarrow, @)$ and the bounded fragment.

3.2 Generated back-and-forth systems

We now turn to the problem of providing semantic characterizations of $\mathcal{H}(\downarrow, @)$ (or equivalently, of the bounded fragment). In this section we shall adopt an essentially first-order approach: we define *generated back-and-forth systems*, basically a restricted form of Ehrenfeucht-Fraïssé game, and link it to the concept of *generated submodels*.

Definition 3.5 (Partial Isomorphism)
Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. A function h from a subset of M to a subset of N is called a *partial isomorphism* if

1. h is a bijection;
2. for all x in the domain of h, for all $a \in \text{PROP} \cup \text{NOM}$, $x \in V^{\mathfrak{M}}(a)$ iff $h(x) \in V^{\mathfrak{N}}(a)$;
3. for all x, y in the domain of h, $R^{\mathfrak{M}}xy$ iff $R^{\mathfrak{N}}h(x)h(y)$.

Generated back-and-forth systems

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two first-order models in the hybrid signature. A generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty family F of finite partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the following two extension rules:

($\Diamond$-extension)
- (forth) if $h \in F$, x in its domain and $R^{\mathfrak{M}}xy$, then $h \cup \{\langle y, y' \rangle\} \in F$ for some $y' \in N$.
- (back) if $h \in F$, x in its range and $R^{\mathfrak{N}}xy$, then $h \cup \{\langle y', y \rangle\} \in F$ for some $y' \in M$.

(nominal extension)
- (forth) if $h \in F$ and there exists an $x \in M$ such that $V^{\mathfrak{M}}(i) = \{x\}$ for some nominal i, then there exists an $x' \in N$ such that $h \cup \{\langle x, x' \rangle\} \in F$.
- (back) a similar condition backwards.

Let $m \in M$ and $n \in N$. Then $(\mathfrak{M}, m) \equiv_R (\mathfrak{N}, n)$ means that there exists a generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$ which connects m and n. More generally, let $\bar{m}$ ($\bar{n}$) be a finite M-tuple (N-tuple). Then $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$ means that there exists a generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$, containing a partial isomorphism which sends $m(i)$ to $n(i)$.

Note how closely this definition follows the familiar one from first-order logic (cf. e.g., [Hod93]). In fact, if we think of such a system as describing an Ehrenfeucht-Fraïssé game, then the sole difference is that in the "generated back-and-forth game" the universal player (∀belard) must choose his moves from R-successors or worlds named by a nominal, whereas he can play whatever he likes in the full first-order game. Unsurprisingly, restricting the play to *accessible* worlds means that generated back-and-forth systems and *generated submodels* are closely connected.

Definition 3.6 (Generated Submodel)
Let $\mathfrak{M} = \langle M, R, V\rangle$ be a hybrid model and $S \subseteq M$. Let NOM denote the subset of M whose elements are in the interpretation of some nominal. The *submodel of* $\mathfrak{M}$ *generated by* S is the substructure of $\mathfrak{M}$ with domain $\{m \in M \mid \exists s \in S \cup \mathrm{NOM}(R^* sm)\}$ (R^* is the reflexive and transitive closure of R). This is also called the S-generated submodel of $\mathfrak{M}$.

Note that if $NOM = \emptyset$, we obtain the familiar modal notion of a generated submodel; and that if in addition S is a singleton set, we have the usual modal notion of a point-generated (or rooted) submodel.

We now define two notions of invariance. The first is taken from [Ben83]. A first-order formula $\varphi(\bar{x})$ in free variables $\bar{x}$ in a signature with one binary relation R, unary predicates and constants (and equality) is *invariant for generated submodels* if for all models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{M}', \bar{m})$ such that $\mathfrak{M}'$ is the $\bar{m}$-generated submodel of $\mathfrak{M}$,

$$\mathfrak{M} \models \varphi(\bar{m}) \text{ if and only if } \mathfrak{M}' \models \varphi(\bar{m}).$$

In a similar spirit, we shall say that a first-order formula $\varphi(\bar{x})$ in the same signature is invariant for generated back-and-forth systems if for all models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$, $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$ implies

$$\mathfrak{M} \models \varphi(\bar{m}) \text{ if and only if } \mathfrak{N} \models \varphi(\bar{n}).$$

Theorem 3.7
Let $\varphi(\bar{x})$ be a first-order formula in the hybrid signature. Then the following are equivalent.

i. $\varphi(\bar{x})$ is equivalent to a formula in the bounded fragment.
ii. $\varphi(\bar{x})$ is invariant for generated submodels.
iii. $\varphi(\bar{x})$ is invariant for generated back-and-forth systems.

Proof.
i. ⇒ *ii.* is obvious.

ii. ⇒ *iii.* First note that $\varphi(\bar{x})$ is invariant for generated submodels if and only if $\neg\varphi(\bar{x})$ is. Now suppose $\varphi(\bar{x})$ is invariant for generated submodels but *not* preserved under generated back-and-forth systems. Then we have models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$, a generated back-and-forth system linking $\bar{m}$ and $\bar{n}$, and $\mathfrak{M} \models \varphi(\bar{m})$ while $\mathfrak{N} \models \neg\varphi(\bar{n})$.

Let $\mathfrak{M}'$ ($\mathfrak{N}'$) be the $\bar{m}$- ($\bar{n}$) generated submodel of $\mathfrak{M}$ ($\mathfrak{N}$). Then still $\mathfrak{M}' \models \varphi(\bar{m})$ and $\mathfrak{N}' \models \neg\varphi(\bar{n})$, by invariance. Clearly $(\mathfrak{M}', m) \equiv_R (\mathfrak{N}', n)$. But then $(\mathfrak{M}', m)$ and $(\mathfrak{N}', n)$ have the same first-order theory by the following argument. Because $(\mathfrak{M}', m) \equiv_R (\mathfrak{N}', n)$ holds, ∃loise has a winning strategy in all games where ∀belard only plays *immediate* R-successors or points named by a nominal. But since the models are generated, in the first-order back-and-forth game he can only play worlds which are accessible by a finite R-transition from either the root or one of the named worlds. But then she can compute a winning answer for the classic Ehrenfeucht-Fraïssé from her winning generated back-and-forth strategy. This contradicts the claim that $\mathfrak{M}' \models \varphi(\bar{m})$ and $\mathfrak{N}' \models \neg\varphi(\bar{n})$.

iii. $\Rightarrow$ *i.* We use a van Benthem style diagram-chasing argument [Ben96]. We only provide the outline. Let $\varphi(\bar{x})$ be as in the hypothesis and $BC(\varphi(\bar{x}))$ the bounded consequences of $\varphi(\bar{x})$ (that is, the consequences of $\varphi(\bar{x})$ that belong to the bounded fragment). We will show that $BC(\varphi(\bar{x})) \models \varphi(\bar{x})$, from which the result follows by compactness. (In this notation we interpret the $\bar{x}$ as constants, or equivalently we use the local version of first-order consequence, cf. [End72].)

If $BC(\varphi(\bar{x}))$ is inconsistent we are done. Otherwise, let $\mathfrak{M}, \bar{m}$ be a model of $BC(\varphi(\bar{x}))$ and $\mathfrak{N}, \bar{n}$ be a model of $\varphi(\bar{x})$ together with the bounded theory of $\mathfrak{M}, \bar{m}$. (Such a model can easily be shown to exist.) Take ω-saturated extensions $\mathfrak{M}^+, \bar{m}$ and $\mathfrak{N}^+, \bar{n}$. Create a family F of finite functions between M^+ and N^+ as follows: $f : \bar{x} \to \bar{y}$ is in F iff $\mathfrak{M}^+, \bar{x}$ and $\mathfrak{N}^+, \bar{y}$ make the same bounded formulas true. It is easy to show that F is a generated back and forth system linking $\bar{m}$ and $\bar{n}$. Now we can start diagram chasing: $\mathfrak{N} \models \varphi(\bar{n})$ then (by elementary extension) $\mathfrak{N}^+ \models \varphi(\bar{n})$, then (by invariance) $\mathfrak{M}^+ \models \varphi(\bar{m})$, then (passing to an elementary submodel) $\mathfrak{M} \models \varphi(\bar{m})$ as desired. ∎

3.3 Hybrid bisimulations

We have just seen that by weakening the notion of an Ehrenfeucht-Fraïssé game we can link the bounded fragment (and hence $\mathcal{H}(\downarrow, @)$) with generated submodels. But in spite of its binding apparatus, $\mathcal{H}(\downarrow, @)$ has a distinctly modal flavor. Is it not also possible to strengthen the notion of *bisimulation* (the standard notion of equivalence between models used in modal logic) with clauses for $\downarrow$ and @, and so characterize $\mathcal{H}(\downarrow, @)$ in intrinsically modal terms? That's what we will do in this section. The approach has an advantage over the use of generated back-and-forth systems: preservation results can be easily obtained for reducts as well.

Recall that for ordinary propositional modal logics, bisimulations are non-empty binary relations linking the domains of models, with the restriction that only worlds with identical atomic information and matching accessibility relations should be connected (see Definition 3.7 [Ben83]; here bisimulations are called p-relations). Now, if we want to extend this notion to $\mathcal{H}(\downarrow, @)$, we need to take care of assignments to world variables as well. To this end, hybrid bisimulations will not simply link worlds, rather they will link pairs $(\bar{m}, m)$, where m is a world and $\bar{m}$ is an assignment. We start by defining k-bisimulations, which are the correct notion of bisimulation for formulas φ such that $\mathrm{WVAR}(\varphi) \subseteq \{x_1, \ldots, x_k\}$.

k-bisimulation

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. Let $\overset{k}{\sim}$ be a binary relation between ${}^kM \times M$ and ${}^kN \times N$. So $\overset{k}{\sim}$ relates tuples $((m_1, \ldots, m_k), m)$ with tuples $((n_1, \ldots, n_k), n)$. We write these tuples as $(\bar{m}, m)$. Notice that $\bar{m}$ can be seen as an assignment over $(x_1, \ldots, x_k)$. A non-empty relation $\overset{k}{\sim}$ is called a *k-bisimulation* if it satisfies the following properties

(prop) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then $m \in V^{\mathfrak{M}}(a)$ iff $n \in V^{\mathfrak{N}}(a)$, for $a \in \mathrm{PROP} \cup \mathrm{NOM}$.

(var) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for all $j \leq k$, $m_j = m$ iff $n_j = n$.

(forth) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$ and $R^{\mathfrak{M}}mm'$, then there exists an $n' \in N$ such that $R^{\mathfrak{N}}nn'$

and $(\bar{m}, m') \overset{k}{\sim} (\bar{n}, n')$.

(back) A similar condition from $\mathfrak{N}$ to $\mathfrak{M}$.

(@) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for every nominal $i \in$ NOM, if $m' \in V^{\mathfrak{M}}(i)$ and $n' \in V^{\mathfrak{N}}(i)$ then $(\bar{m}, m') \overset{k}{\sim} (\bar{n}, n')$, and for every $j \leq k$, $(\bar{m}, m_j) \overset{k}{\sim} (\bar{n}, n_j)$

(↓) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for every $j \leq k$, $(\bar{m}^j_m, m) \overset{k}{\sim} (\bar{n}^j_n, n)$.

Note that since ↓ and @ are self-dual, we can collapse the back and forth clauses for these modalities into one. We write $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$ if there exists a k-bisimulation between the two models.

To extend the notion to the full language we need to add only one further condition.

ω-bisimulation.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. An ω-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty family of k-bisimulations satisfying the following *storage rule*:

(sto) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then $(\bar{m} * m, m) \overset{k+1}{\sim} (\bar{n} * n, n)$.

Here and elsewhere, $\bar{m} * m$ denotes the tuple obtained from concatenating $\bar{m}$ and m. Let $\bar{m}$ ($\bar{n}$) be an M-tuple (N-tuple). Then $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$ means that there exists an ω-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that $(\bar{m}, m(0)) \overset{k}{\sim} (\bar{n}, n(0))$, for k the length of $\bar{m}$.

Some remarks. First, k and ω-bisimulations can be restricted to a given set of propositional variables and nominals PROP ∪ NOM by restricting **(prop)** and **(@)** accordingly. Second, the modular definition of k and ω-bisimulation will lead to results for reducts of the language as well. For instance if we delete ↓ from the language, we just delete the **(↓)** clause from the definition of bisimulation and we obtain the appropriate notion for $\mathcal{H}(@)$. Of course, if we delete the variables from the language, we don't need the assignment tuples anymore, and the bisimulation becomes just a relation between worlds, as usual. Then for the language without ↓, @ and variables, the standard definition of bisimulation applies (the condition **(prop)** takes care of the nominals). If we add @ to this language, we just have to add the following clause

(@′) For all nominals i, if $V^{\mathfrak{M}}(i) = \{m\}$ and $V^{\mathfrak{N}}(i) = \{n\}$, then $m \sim n$.

Preservation results for all these alternatives can be given (the required proofs follow much the same lines as the proofs below).

The first important fact about hybrid bisimulations is that they preserve truth:

Proposition 3.8

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models, $m \in M$, $n \in N$. Then,

i. If $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$, with $\overset{k}{\sim}$ over a given set PROP ∪ NOM, then for all formulas φ over the signature $\langle \text{PROP}, \text{NOM}, \{x_1, \ldots, x_k\}\rangle$, $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$ implies $\mathfrak{M}, \bar{m}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash \varphi$.

ii. If $(\mathfrak{M}, m) \overset{\omega}{\sim} (\mathfrak{N}, n)$, with $\overset{\omega}{\sim}$ over a given set PROP ∪ NOM, then for all *sentences* φ over the signature $\langle \text{PROP}, \text{NOM}, \text{WVAR}\rangle$, $\mathfrak{M}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, n \Vdash \varphi$. (Recall that for sentences the choice of assignment is irrelevant.)

PROOF.

i. By a straightforward inductive argument.

ii. Let $(\mathfrak{M}, m) \overset{\omega}{\sim} (\mathfrak{N}, n)$ and let φ be a hybrid sentence. Then it contains variables (after renaming) say $\{x_1, \ldots, x_k\}$. We have $(\langle m \rangle, m) \overset{1}{\sim} (\langle n \rangle, n)$, so $k-1$ applications of the storage rule give us $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, where $\bar{m}$ is a k-tuple consisting of m's and similarly for $\bar{n}$. But then, by *i.*, $\mathfrak{M}, \bar{m}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash \varphi$, whence since φ is a sentence $\mathfrak{M}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, n \Vdash \varphi$. ∎

The notion of k-bisimulation has a distinct modal flavor. But a very first-order notion is hidden inside: partial isomorphism.

PROPOSITION 3.9

Let $k \geq 2$, and let $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$. If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then the function f defined as $f(m) = n$ and $f(m(i)) = n(i)$ is a partial isomorphism between $\{m(1), \ldots, m(k), m\}$ and $\{n(1), \ldots, n(k), n\}$.

PROOF. The map f is a bijection by **(var)** and **(@)**. By **(prop)** and **(@)**, f preserves nominals and propositional variables. To see that it preserves the accessibility relation suppose $R^{\mathfrak{M}}xy$. There are three cases.

(Case 1: $x = m,\ y = m_i$.) Then by **(forth)** there exists an n' such that $R^{\mathfrak{N}}nn'$ and $(\bar{m}, m_i) \overset{k}{\sim} (\bar{n}, n')$. But $\bar{m}(i) = m_i$, so by **(var)**, $n' = \bar{n}(i)$, whence $R^{\mathfrak{N}}nf(m(i))$.

(Case 2: $x = m_i,\ y = m$.) Let $j \neq i$. Such a j exists because we assumed that $k \geq 2$. By **(↓)**, $(\bar{m}^j_m, m) \overset{k}{\sim} (\bar{n}^j_n, n)$. Then by **(@)**, $(\bar{m}^j_m, m_i) \overset{k}{\sim} (\bar{n}^j_n, n_i)$. Now continue as in case 1.

(Case 3: $x = m_i,\ y = m_j$.) By **(@)**, $(\bar{m}, m_i) \overset{k}{\sim} (\bar{n}, n_i)$. Now continue as in case 1. Thus $R^{\mathfrak{M}}xy$ implies $R^{\mathfrak{N}}f(x)f(y)$. For the other direction use **(back)** in the same way. ∎

Note that the condition $k \geq 2$ is crucial. We use it together with **(↓)** to store the information about m. Of course in a model where Rm_im holds, we have $\bar{m}, m \Vdash \downarrow x_j.@_{x_i}\Diamond x_j$.

Thus there is a clear link between our earlier work on generated back-and-forth systems, and the next theorem shouldn't come as a surprise:

THEOREM 3.10

Let $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$ be two models. Then the following are equivalent:

i. $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$.

ii. $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$.

PROOF.

i. ⇒ *ii.* Let $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$. Define a family F of maps as follows: $f \in F$ if there exists $(\bar{x}, x') \overset{k}{\sim} (\bar{y}, y')$ and f is defined as in Proposition 3.9.

Clearly $\bar{m}$ and $\bar{n}$ are connected by a map. By Proposition 3.9 all maps are partial isomorphisms. We show the forth side of **(nominal extension)**; all other conditions have a similar proof. Suppose $f \in F$ and $\mathbf{z} \in M$ and $V^{\mathfrak{M}}(i) = \{\mathbf{z}\}$, for some nominal i. Then for some $\bar{x}$, x, $\bar{y}$, y, $(\bar{x}, x') \overset{k}{\sim} (\bar{y}, y')$ by definition of F. Then $(\bar{x} * x', x') \overset{k+1}{\sim} (\bar{y} * y', y')$ by **(sto)**. But then by **(@)**, $(\bar{x} * x', \mathbf{z}) \overset{k+1}{\sim} (\bar{y} * y', \mathbf{z}')$ for $V^{\mathfrak{N}}(i) = \{\mathbf{z}'\}$. Thus the wanted extension is in F.

ii. $\Rightarrow$ *i.* Let $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$. We define an ω-bisimulation in the obvious way: for any $f \in F$, for any k, for any tuple $\bar{m}$ in the k-th power of the domain of f and for any m in the domain of f, we set $(\bar{m}, m) \overset{k}{\sim} (f\bar{m}, f(m))$. It is easy to check that this is an ω-bisimulation. ∎

It is possible to prove a direct characterization result for $\mathcal{H}(\downarrow, @)$ in terms of invariance for k-bisimulations, using again a diagram chasing argument. We're not going to do this here since in the next section we shall take a detour via the bounded fragment to reach the same result. It is also possible to develop k-pebble versions of generated back-and-forth systems; this notion takes the exact number of variables used in formulas into account. It is not difficult to see that $k+1$-pebble generated back-and-forth systems correspond to k-bisimulations, but we won't follow up these matters here.

4 Harvest

It is time to draw together the threads developed in the previous section. First we note their consequences for $\mathcal{H}(\downarrow, @)$ expressivity over models. Then we note the consequences for frames and what this tells us about hybrid completeness.

4.1 Expressivity over models

We have the following five-fold characterization of $\mathcal{H}(\downarrow, @)$:

Theorem 4.1
Let $\varphi(\bar{x})$ be a first-order formula in the hybrid signature (with equality). Then the following are equivalent.

i. $\varphi(\bar{x})$ is equivalent to the standard translation of a $\mathcal{H}(\downarrow, @)$ formula.
ii. $\varphi(\bar{x})$ is invariant for generated submodels.
iii. $\varphi(\bar{x})$ is invariant for generated back-and-forth systems.
iv. $\varphi(\bar{x})$ is invariant for ω-bisimulation.
v. $\varphi(\bar{x})$ is equivalent to a formula in the bounded fragment of first-order logic.

Proof. By Corollary 3.4, Theorem 3.7 and Proposition 3.8. ∎

But these have obvious consequences for the ordinary modal correspondence language. In particular, if we consider *nominal-free* hybrid sentences, then we obtain a five-fold characterization of the fragment of first-order logic in the classical modal signature which is invariant for generated submodels:

Corollary 4.2
Let $\varphi(x)$ be a first-order formula in the modal signature with equality. Then the following are equivalent.

i. $\varphi(x)$ is equivalent to the standard translation of a nominal-free $\mathcal{H}(\downarrow, @)$ sentence.
ii. $\varphi(x)$ is invariant for generated submodels (now in the standard modal sense).
iii. $\varphi(x)$ is invariant for R-generated back-and-forth systems, where an R-generated back-and-forth system is a back-and-forth system satisfying only the ***◇-extension*** rule.
iv. $\varphi(x)$ is invariant for ω-bisimulation.

v. $\varphi(x)$ is equivalent to a formula in the bounded fragment of first-order logic without constants.

4.2 Frames and completeness

Recall that a frame $\mathfrak{F}$ is a pair $\langle W, R\rangle$ (that is, a model without a valuation). Since the late 1950s, one of the central topics in modal logic has been linking modal formulas to properties of frames and investigating when they give rise to complete axiomatizations for the frame classes they define. The work of the previous section easily yields a characterization of the frame-defining abilities of pure nominal-free sentences. Moreover, the axiomatic investigations of [BT99] (and indeed, the tableaux-based investigations of [Bla98]) show that there is a perfect match between definability and completeness for pure nominal-free sentences. By combining these results we obtain matching definability and completeness results for a wide range of first-order definable frame classes.

In modal correspondence theory, the first-order language (with equality) over the signature consisting simply of a binary symbol R is called the (first-order) *frame language*. We shall call a formula φ in the frame language containing exactly one free variable a *frame condition*. The class of frames defined by a frame condition $\varphi(x)$ is the class in which the universal closure $\forall x\varphi(x)$ is true; we call this class $\textsc{frames}(\forall x.\varphi(x))$.

Before proceeding further, two simple observations are in order. First, note that if we apply the standard translation ST to a pure nominal-free sentence α , then $ST(\alpha)$ is a frame condition with free-variable x. Furthermore, note that for any frame $\mathfrak{F} = \langle W, R\rangle$ we have that $\mathfrak{F} \Vdash \alpha$ iff $\mathfrak{F} \Vdash \forall x.ST(\alpha)$; this is an immediate consequence of the definition of frame validity (see Definition 2.4).

Theorem 4.3
Let $\mathsf{K}[\mathcal{H}(\downarrow, @)]$ be the axiomatization given in Section 2, and for any hybrid sentence α let $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ be the system obtained by adding α as an additional axiom. Then, if $\varphi(x)$ is a frame condition and $\varphi(x)$ is invariant under generated submodels (in the usual modal sense) we have that:

i. If $\varphi(x)$ is in the bounded fragment, then the pure nominal free sentence $\downarrow x.HT(\varphi(x))$ defines $\textsc{frames}(\forall x.\varphi(x))$. Moreover, $\mathsf{K}[\mathcal{H}(\downarrow, @)]+ \downarrow x.HT(\varphi(x))$ is strongly complete with respect to $\textsc{frames}(\forall x.\varphi(x))$.
ii. If $\varphi(x)$ is not in the bounded fragment, there is a pure nominal free sentence α such that α defines $\textsc{frames}(\forall x.\varphi(x))$, and $ST(\alpha)$ is equivalent to $\varphi(x)$. Moreover, $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ is strongly complete with respect to $\textsc{frames}(\forall x.\varphi(x))$.

Conversely, if α is pure nominal-free sentence, then α defines $\textsc{frames}(\forall x.ST(\alpha(x)))$, and $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ is complete with respect to $\textsc{frames}(\forall x.ST(\alpha(x)))$.

Proof. The converse condition was proved in [BT98], so let's examine the other direction.

For item *i.*, we first remark that as $\varphi(x)$ belongs to the *frame* language, it contains no unary predicate symbols, hence $HT(\varphi(x))$ is a *pure* formula; that $\downarrow x.HT(\varphi(x))$ is a pure nominal-free sentence is thus clear. Now, by Corollary 3.3, for any model

$\mathfrak{M} = (\mathfrak{F}, V)$ and any $m \in M$,

$$(\mathfrak{F}, V) \models \varphi(m) \text{ iff } (\mathfrak{F}, V), m \Vdash \downarrow x.HT(\varphi).$$

But this means that

$$(\mathfrak{F}, V) \models \forall x.\varphi \text{ iff } (\mathfrak{F}, V) \Vdash \downarrow x.HT(\varphi).$$

But as $\varphi(x)$ contains no unary predicate symbols (and $\downarrow x.HT(\varphi)$ no propositional variables) V is irrelevant, and hence

$$\mathfrak{F} \models \forall x.\varphi(x) \text{ iff } \mathfrak{F} \Vdash \downarrow x.HT(\varphi).$$

But this means that $\downarrow x.HT(\varphi(x))$ defines $\textsc{frames}(\forall x.\varphi(x))$. Completeness follows using the arguments of [BT98].

For item *ii.*, we know that $\varphi(x)$ being invariant under generated submodels is equivalent to a formula in the bounded fragment — but is it equivalent to a *frame condition* $\varphi'(x)$? In fact, this can be established by modifying the diagram chasing argument used in the proof of Theorem 3.7. The key point to observe is that instead of showing that $BC(\varphi(x)) \models \varphi(x)$, we can show by the same method that $FC(\varphi(x)) \models \varphi(x)$, where FC are all the frame conditions implied by $\varphi(x)$. Thus there is an equivalent frame condition $\varphi'(x)$, and we can take α to be $\downarrow x.HT(\varphi'(x))$. The remainder of the proof is as for item *i.* ∎

Acknowledgments The authors want to thank Johan van Benthem. His suggestion of using the bounded fragment greatly improved the presentation of the results in Section 3. Maarten Marx is supported by the Spinoza Project 'Logic in Action.' This article arose from a visit of Marx to Blackburn partially funded by the Dutch research School in Logic and the 'Internationaliserings'-program of the Netherlands Organization for Scientific Research.

References

[AvBN98] H. Andréka, J. van Benthem, and I. Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 27(3):217–274, 1998.

[Ben83] J. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Naples, 1983.

[Ben96] J. van Benthem. *Exploring Logical Dynamics.* Studies in Logic, Language and Information. CSLI Publications, Stanford, 1996.

[Bla93] P. Blackburn. Nominal tense logic. *Notre Dame Journal of Formal Logic*, 34(1):56–83, 1993.

[Bla98] P. Blackburn. Internalizing labeled deduction. Technical Report CLAUS-Report 102, Computerlinguistik, Universität des Saarlandes, 1998. `http://www.coli.uni-sb.de/cl/claus`.

[BS95] P. Blackburn and J. Seligman. Hybrid languages. *Journal of Logic, Language and Information*, 4(3):251–272, 1995. Special issue on decompositions of first-order logic.

[BS98] P. Blackburn and J. Seligman. What are hybrid languages? In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 1, pages 41–62. CSLI Publications, Stanford University, 1998.

[BT98] P. Blackburn and M. Tzakova. Hybrid languages and temporal logics (full version). Technical Report CLAUS-Report 96, Computerlinguistik, Universität des Saarlandes, 1998. `http://www.coli.uni-sb.de/cl/claus`.

[BT99] P. Blackburn and M. Tzakova. Hybrid languages and temporal logics. *Logic Journal of the IGPL*, 7(1):27–54, 1999.

[Bul70] R. Bull. An approach to tense logic. *Theoria*, 36:282–300, 1970.

[CK90] C. Chang and H. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[Com85] B. Comrie. *Tense*. Cambridge University Press, 1985.

[End72] H. Enderton. *A mathematical introduction to logic*. Academic Press, New York, 1972.

[GG93] G. Gargov and V. Goranko. Modal logic with names. *Journal of Philosophical Logic*, pages 607–636, 1993.

[Gor96] V. Goranko. Hierarchies of modal and temporal logics with reference pointers. *Journal of Logic, Language and Information*, 5(1):1–24, 1996.

[GP92] V. Goranko and S. Passy. Using the universal modality: gains and questions. *Journal of Logic and Computation*, 2:5–30, 1992.

[Hod93] W. Hodges. *Model theory*. Cambridge University Press, Cambridge, 1993.

[MV97] M. Marx and Y. Venema. *Multi-dimensional Modal Logic*. Applied Logic Series. Kluwer Academic Publishers, 1997.

[Pri67] A. Prior. *Past, Present and Future*. Clarendon Press, Oxford, 1967.

[PT85] S. Passy and T. Tinchev. Quantifiers in combinatory PDL: completeness, definability, incompleteness. In *Fundamentals of computation theory (Cottbus, 1985)*, pages 512–519. Springer, Berlin, 1985.

[PT91] S. Passy and T. Tinchev. An essay in combinatory dynamic logic. *Information and Computation*, 93(2):263–332, 1991.

[Rei47] H. Reichenbach. *Elements of Symbolic Logic*. Random House, New York, 1947.

[Sel91] J. Seligman. A cut-free sequent calculus for elementary situated reasoning. Technical Report HCRC/RP-22, HCRC, Edinburgh, 1991.

[Sel97] J. Seligman. The logic of correct description. In M. de Rijke, editor, *Advances in Intensional Logic*, pages 107–135. Kluwer, 1997.

[Tza98] M. Tzakova. Tableaux calculi for hybrid logics. Manuscript. Will be made available at `http://www.mpi-sb.mpg.de/~tzakova/`, 1998.

[Ven94] Y. Venema. A modal logic for quantification and substitution. *Bulletin of the IGPL*, 2(1):31–45, 1994.

Received March 4 1999