Hybrid Logics

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Stanford

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- Article on "Hybrid Logics" at the Stanford Encyclopedia of Philosophy, `http://plato.stanford.edu/entries/logic-hybrid`

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Syntax: Propositional Logic + modalities

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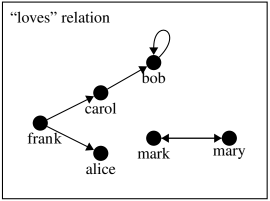

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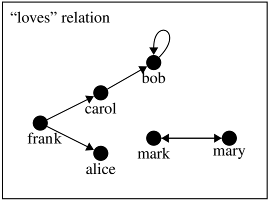

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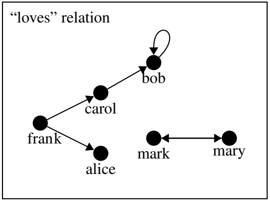

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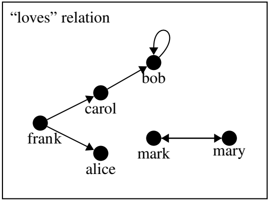

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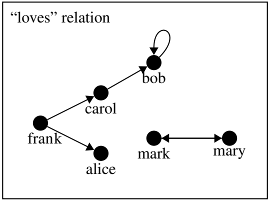
- MLs can be (usually) thought of as fragments of FO
- Modal logics are (usually) decidable
  - SAT for the basic modal logic is PSpace-complete

The Limits of Modal Expressivity

Some properties can't be expressed in the basic modal language…



Query: “Does Frank love Alice?”

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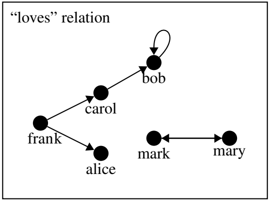

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- What do we need?
  - constants
  - identity

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  - Although states are crucial to modal semantics, nothing in modal syntax can talk about them.
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  - A representation problem: for some applications modal logic is not adequate as a representation formalism, and
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  - A representation problem: for some applications modal logic is not adequate as a representation formalism, and
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- These limitations motivated the work on Hybrid Logics.

The Basic Recipe: $\mathcal{H}(@)$

basic modal logic

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  In particular $@_ij$ says that i and j denote the same point in the model (i.e., $i = j$).

The Hybrid Logic $\mathcal{H}(@)$

Syntax:

$$\text{FORM} := p \mid i \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi \mid @_i\varphi, i \in \text{NOM}$$

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Semantics: Use Kripke models $\langle W, R, V\rangle$, add an assignment function $g : \text{NOM} \to W$. We call $\langle W, R, V\rangle, g, w$ a hybrid pointed model.

Define
$$\mathcal{M}, g, w \models i \quad \text{iff} \quad w = g(i) \text{ for } i \in \text{NOM}$$

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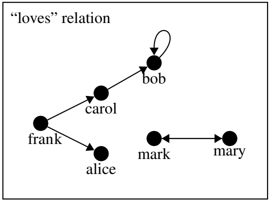
$$\begin{array}{rcl} \mathcal{M}, g, w \models i & \text{iff} & w = g(i) \text{ for } i \in \mathsf{NOM} \\ \mathcal{M}, g, w \models @_i\varphi & \text{iff} & \mathcal{M}, g, g(i) \models \varphi \end{array}$$

The Expressive Power of $\mathcal{H}(@)$



Query: "Are there two people who loves each other?"

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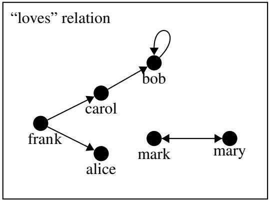

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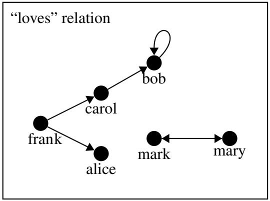

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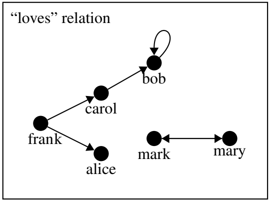

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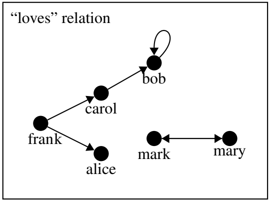

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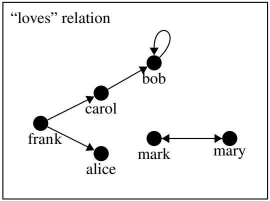

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$$@_i j \rightarrow (@_i \varphi \leftrightarrow @_j \varphi)$$

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- Consider the formula $@_i \Diamond j \wedge @_j \varphi \rightarrow @_i \Diamond \varphi$. Valid?

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Theorem: $\mathcal{M}, g, w \models \varphi$ iff $\mathcal{M}', f[x \mapsto w] \models \mathsf{ST}_x(\varphi)$.

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- Introduce the basic hybrid logics $\mathcal{H}(@)$.
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Bisimulations for $\mathcal{H}@$

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- The converse holds on finite-image models (as with BML).
- A "van Benthem Characterization Theorem" holds using the notion of bisimulation for $\mathcal{H}(@)$.

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- Think about it for next class.

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- What about the basic modal language?

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- They will help us keep track of what is going on in each point in our model.

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- To think: what is the status of labeled formulas $s{:}\varphi$ and accessibility statements sRt?

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The Complete Cast, plus an Example

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$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c} s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\ s{:}\Diamond p \\ s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\ s{:}\neg\Diamond\neg q \\ s{:}\neg\Diamond(p \wedge q) \end{array}$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q)))$$
$$s{:}\Diamond p$$
$$s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q)$$
$$s{:}\neg\Diamond\neg q$$
$$s{:}\neg\Diamond(p \wedge q)$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\begin{array}{c} s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\ s{:}\Diamond p \\ s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\ s{:}\neg\Diamond\neg q \\ s{:}\neg\Diamond(p \wedge q) \\ sRt \\ t{:}p \end{array}$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c} s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\ s{:}\Diamond p \\ s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\ s{:}\neg\Diamond\neg q \\ s{:}\neg\Diamond(p \wedge q) \\ sRt \\ t{:}p \end{array}$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \land \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \land \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$s{:}(\Diamond p \land (\neg\Diamond\neg q \land \neg\Diamond(p \land q)))$$
$$s{:}\Diamond p$$
$$s{:}\neg\Diamond\neg q \land \neg\Diamond(p \land q)$$
$$s{:}\neg\Diamond\neg q$$
$$s{:}\neg\Diamond(p \land q)$$
$$sRt$$
$$t{:}p$$
$$t{:}\neg\neg q$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q
\end{array}$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

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$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

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s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q
\end{array}$$

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$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q
\end{array}$$

The Complete Cast, plus an Example

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$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q)
\end{array}$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q)
\end{array}$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \land \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \land \psi)}{s{:}\neg\varphi \qquad\qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$s{:}(\Diamond p \land (\neg\Diamond\neg q \land \neg\Diamond(p \land q)))$$
$$s{:}\Diamond p$$
$$s{:}\neg\Diamond\neg q \land \neg\Diamond(p \land q)$$
$$s{:}\neg\Diamond\neg q$$
$$s{:}\neg\Diamond(p \land q)$$
$$sRt$$
$$t{:}p$$
$$t{:}\neg\neg q$$
$$t{:}q$$
$$t{:}\neg(p \land q)$$
$$t{:}\neg p \qquad\qquad t{:}\neg q$$

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad\qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q) \\
t{:}\neg p \qquad\qquad t{:}\neg q
\end{array}$$

closed closed

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad\qquad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q) \\
t{:}\neg p \qquad\qquad t{:}\neg q \\
\\
\text{closed} \qquad\qquad \text{closed}
\end{array}$$

- Which are the similarities/differences with tableaux for PL?

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\begin{array}{c}
s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q) \\
t{:}\neg p \qquad t{:}\neg q \\
\text{closed} \qquad \text{closed}
\end{array}$$

- Which are the similarities/differences with tableaux for PL?
- How do we know that we got it right?

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q)))$

$s{:}\Diamond p$

$s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q)$

$s{:}\neg\Diamond\neg q$

$s{:}\neg\Diamond(p \wedge q)$

sRt

$t{:}p$

$t{:}\neg\neg q$

$t{:}q$

$t{:}\neg(p \wedge q)$

$t{:}\neg p$ | $t{:}\neg q$

closed | closed

- Which are the similarities/differences with tableaux for PL?
- How do we know that we got it right?
- What can we learn from the calculus?

A Closer Look

A Closer Look

- Which similarities / differences with tableaux for PL?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad\quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?
  - Is it a good algorithm?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}} \text{ for } t \text{ a new label}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?
  - Is it a good algorithm?
  - Does it terminate?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

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A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?
  - Is it a good algorithm?
  - Does it terminate?
- Did we get it right?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
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$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}} \text{ for } t \text{ a new label}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

- Did we get it right?
  - Did we get it right in the PL case, to start with?! Consider the rule:

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad \begin{array}{c} s{:}\varphi \\ s{:}\neg\psi \end{array}}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?
  - Is it a good algorithm?
  - Does it terminate?
- Did we get it right?
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  - We should prove soundness and completeness

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}} \text{ for } t \text{ a new label}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad \begin{array}{c} s{:}\varphi \\ s{:}\neg\psi \end{array}}$$

A Closer Look

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  - Is this an algorithm?
  - Is it a good algorithm?
  - Does it terminate?
- Did we get it right?
  - Did we get it right in the PL case, to start with?! Consider the rule:
  - We should prove soundness and completeness
- What can we learn from the calculus?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}} \text{ for } t \text{ a new label}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad \begin{array}{c} s{:}\varphi \\ s{:}\neg\psi \end{array}}$$

A Closer Look

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  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?
  - Is it a good algorithm?
  - Does it terminate?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}} \text{ for } t \text{ a new label}$$

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

- Did we get it right?
  - Did we get it right in the PL case, to start with?! Consider the rule:

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad \mid \quad \begin{array}{c} s{:}\varphi \\ s{:}\neg\psi \end{array}}$$

  - We should prove soundness and completeness
- What can we learn from the calculus?
  - Something about models!

Tree Models

Tree Models

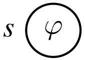
- Let us see the tableux proof we did before again, for the formula

$$\varphi = s{:}(\Diamond p \land (\neg\Diamond\neg q \land \neg\Diamond(p \land q)))$$

Tree Models

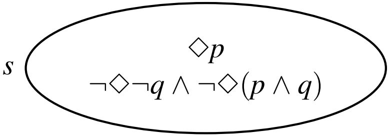
- Let us see the tableux proof we did before again, for the formula

$$\varphi = s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q)))$$



Tree Models

- Let us see the tableux proof we did before again, for the formula



$$\varphi = s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q)))$$
$$s{:}\Diamond p$$
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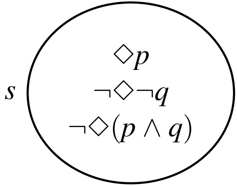

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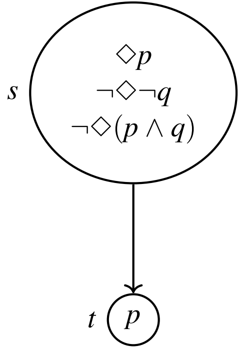

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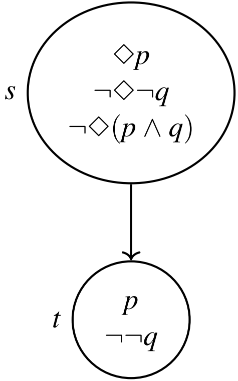

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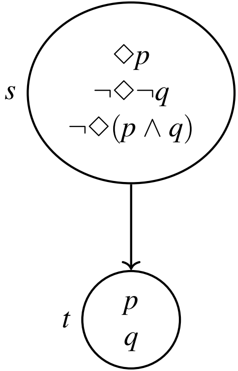

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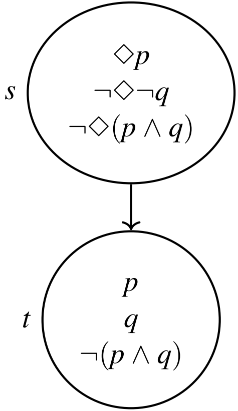

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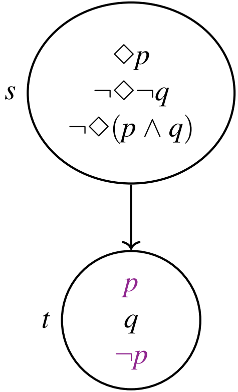

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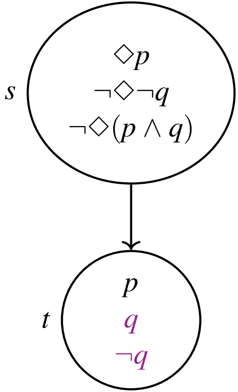

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Theorem: A formula in the $\langle R\rangle$-language is satisfialle if and only if it is satisfiable in a finite, tree relational structure.

Soundness and Completeness

- We would like to verify that:
  - if the tableau for φ is closed (all tranches contains a class) then φ is UNSAT [Soundness].
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- To show completeness we need to build a model from a saturated, open branch.

Completeness

Theorem. If Γ is a saturated open branch from a tableaux for φ, then φ is SAT.

Proof. Given Γ we define the model $\mathcal{M}_\Gamma = \langle W_\Gamma, R_\Gamma, V_\Gamma \rangle$ where

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Hence, $w{:}\varphi \in \Gamma$ implies $\mathcal{M}_\Gamma, w \models \varphi$.

What we want to cover

- Introduce the basic hybrid logics $\mathcal{H}(@)$.
- Extend the standard translation to $\mathcal{H}(@)$.
- Talk about expressive power. Bisimulation.
- Define a tableax algorithm for satisfiability of formulas in the basic modal logic (BML).
  - How do we check whether a formula has a model?
  - What can we learn from tableaux?
- Transform the tableau for BML into one for $\mathcal{H}(@)$.
- Discuss more expressive hybrid logics.

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- Fact 2: : is self-dual, hence $\neg i{:}\varphi$ is equivalent to $i{:}\neg\varphi$.

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Equality: (Ref) $\dfrac{[i \text{ on branch}]}{i{:}i}$ $\qquad$ (Sym) $\dfrac{i{:}j}{j{:}i}$ $\qquad$ (Cong) $\dfrac{i{:}k \quad j{:}k \quad i{:}\varphi}{j{:}\varphi}$

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Hybrid Termination

- Once nominals and satisfaction operators are introduced, ensuring termination is more difficult.
- An obvious problem is the *Cong* rule: $\dfrac{i{:}k \quad j{:}k \quad i{:}\varphi}{j{:}\varphi}$
- The solution is to impose a "direction": Only one nominal in the equivalence class is saturated. But equality should still be an equivalence, so an irrestricted version for nominals is introduced.

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Bolander, T. and Blackburn, P.
Termination for Hybrid Tableaus
Journal of Logic and Computation, 17, 517-554, 2007.

A Tableau Based Prover for Hybrid Logics

- A sound, complete and terminating calculus has been implemented in the prover HTab
- Available at `https://hackage.haskell.org/package/HTab`.
- Implemented in haskell, with a GPL license. I.e., you can get the code and change it!
- Demo

What we want to cover

- Introduce the basic hybrid logics $\mathcal{H}(@)$.
- Extend the standard translation to $\mathcal{H}(@)$.
- Talk about expressive power. Bisimulation.
- Define a tableax algorithm for satisfiability of formulas in the basic modal logic (BML).
  - How do we check whether a formula has a model?
  - What can we learn from tableaux?
- Transform the tableau for BML into one for $\mathcal{H}(@)$.
- Discuss more expressive hybrid logics.

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We use hybrid models $\langle W, R, V\rangle, g$, but $g : \mathsf{NOM} \cup \mathsf{VAR} \to W$.

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Bad News:

- Satisfiability for $\mathcal{H}(\downarrow)$ (i.e., without nominals and @) is already undecidable.