Proving Equations on Stream GSOS via Bisimulation on Open Terms

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CMCS 2016
April 3, 2016

Outline

No outline... ;)

Operations over streams can be defined by means of stream GSOS specification

$$\langle \Sigma, A, R \rangle$$

- Σ a set of operations.
- A the set of actions that can be executed by the streams.
- R a set of rules, defining the semantic of the operators.

A well-known result:

Bisimilarity is a congruence.

Example (1/2)

- $\Sigma = \{\mathbf{a} \mid a \in A\} \cup \{\text{alt}\}$.
- A a set of actions.
- R a set of rules containing:

$$\frac{}{\mathbf{a} \xrightarrow{a} \mathbf{a}}\ \forall a \in A \qquad \frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\text{alt}(x, y) \xrightarrow{a} \text{alt}(y', x')}\ \forall a, b \in A$$

Example (2/2)

$$\frac{}{\mathbf{a} \xrightarrow{a} \mathbf{a}} \; \forall a \in A \qquad \frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\mathrm{alt}(x, y) \xrightarrow{a} \mathrm{alt}(y', x')} \; \forall a, b \in A$$

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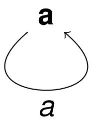

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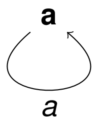

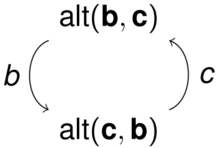

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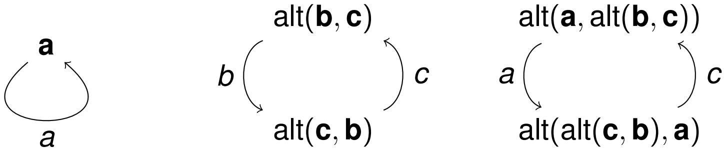

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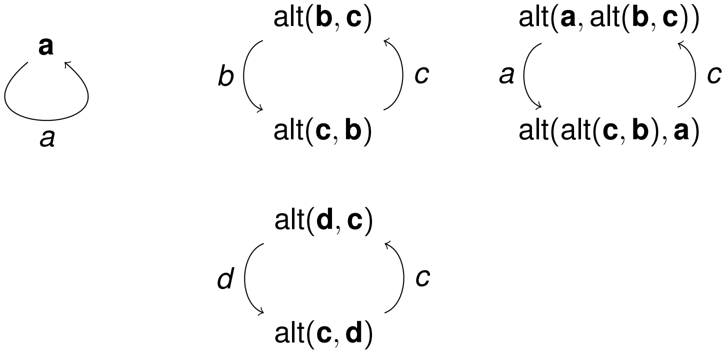

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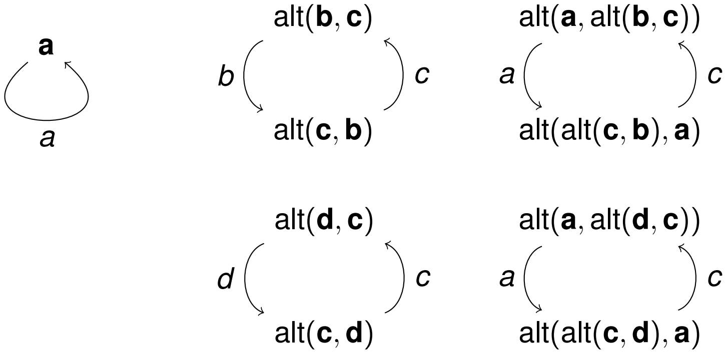
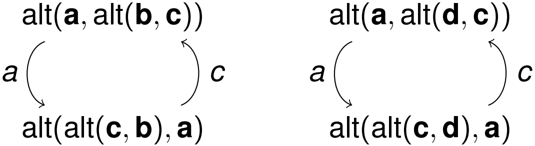


These systems are bisimilar



In general

$$\mathrm{alt}(X, \mathrm{alt}(Y, Z)) \quad \equiv \quad \mathrm{alt}(X, \mathrm{alt}(W, Z))$$

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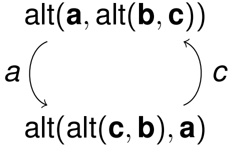

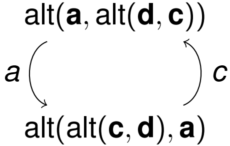

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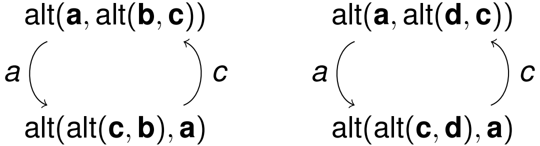

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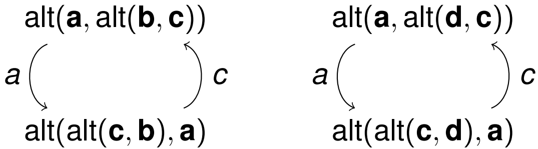

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$$\mathrm{alt}(X, \mathrm{alt}(Y, Z)) \quad \equiv \quad \mathrm{alt}(X, \mathrm{alt}(W, Z))$$

- Q: How do we prove that the equation is sound?
- A: For all possible closed instantiation of X, Y, Z check that both terms are bisimilar.

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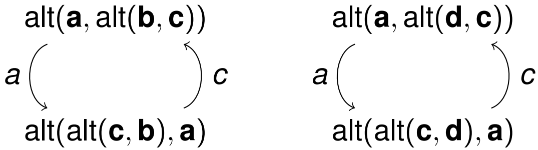

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- Q: How do we prove that the equation is sound?

- A': We define a transition systems over open terms based on the stream system. The equation is sound if both sides of the equation are bisimilar in the new model.

Transition systems over open terms - The intuition

$$\text{alt}(X, t) \text{ with } X \in \mathcal{V}, t \in T_\Sigma\mathcal{V}$$

Recall

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\text{alt}(x, y) \xrightarrow{a} \text{alt}(y', x')} \ \forall a, b \in A$$

Transition systems (TS) over open terms

- Let $\sigma : \mathcal{V} \to A$ be a function that defines the output that can be executed by a variable.
- We consider *Mealy machine* $(T_\Sigma \mathcal{V}, \alpha)$ with
  - inputs in $A^\mathcal{V}$ and
  - outputs in A where

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Specification of TS over open terms

- New rules describing the behavior of the variables.

$$\frac{}{X \xrightarrow{\sigma|\sigma(X)} X}$$

- The original rules are lifted to rules for Mealy machines:

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\mathrm{alt}(x,y) \xrightarrow{a} \mathrm{alt}(y',x')} \quad \rightsquigarrow \quad \frac{x \xrightarrow{\sigma|a} x' \quad y \xrightarrow{\sigma|b} y'}{\mathrm{alt}(x,y) \xrightarrow{\sigma|a} \mathrm{alt}(y',x')}$$

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Example:

Taking into account the rules

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we get for all $\sigma, \sigma' : \mathcal{V} \to A$

$$\mathrm{alt}(Y, Z) \xrightarrow{\sigma|\sigma(Y)} \mathrm{alt}(Z, Y) \qquad \mathrm{alt}(Z, Y) \xrightarrow{\sigma'|\sigma'(Z)} \mathrm{alt}(Y, Z)$$

$$\mathrm{alt}(X, \mathrm{alt}(Y, Z)) \xrightarrow{\sigma|\sigma(X)} \mathrm{alt}(\mathrm{alt}(Z, Y), X) \qquad \mathrm{alt}(\mathrm{alt}(Z, Y), X) \xrightarrow{\sigma'|\sigma'(Z)} \mathrm{alt}(X, \mathrm{alt}(Y, Z))$$

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Soundness of the lifting

Theorem

For all $t_1, t_2 \in T_\Sigma\mathcal{V}$*:* $t_1 \sim_\mathcal{M} t_2$ *iff* $t_1 \equiv t_2$*.*

- The proof uses the results presented in the paper *"Pointwise extensions of GSOS-defined operations"* by Hansen and Klin.

Up-to substitution - Motivation

- Consider over streams the stop process $\underline{\emptyset}$ and the unary operator f defined by:

$$\frac{}{f(x) \xrightarrow{1} f(f(x))}$$

then

$$f(\underline{\emptyset}) \xrightarrow{1} f(f(\underline{\emptyset})) \xrightarrow{1} f(f(f(\underline{\emptyset}))) \xrightarrow{1} f(f(f(f(\underline{\emptyset})))) \xrightarrow{1} \dots$$

- $f(\underline{\emptyset}) \sim_{\mathcal{M}} f(\underline{\emptyset})$
- For all bisimulation over open terms R,

$$(f(\underline{\emptyset}), f(\underline{\emptyset})) \in R \text{ implies } |R| = \infty$$

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Up-to substitution - Motivation

- Having up-to substitution would allow to use the relation

$$R = \{(f(\underline{\emptyset}), f(\underline{\emptyset})), (X, X)\}$$

to prove that $f(\underline{\emptyset}) \sim_{\mathcal{M}} f(\underline{\emptyset})$

- All the other pairs needed by the bisimulation can be obtained as an instantiation of the pair

$$(X, X)$$

First result:

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Up-to substitution is not sound.

From Rules for streams to rules for Mealy machines

- To lift some rules we need to introduce the *buffer operator* ($\triangleright$).
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Without the buffer operator information is lost

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{x \boxplus y \xrightarrow{a+b} x \boxplus y'} \quad \rightsquigarrow \quad \frac{x \xrightarrow{\sigma|a} x' \quad y \xrightarrow{\sigma|b} y'}{x \boxplus y \xrightarrow{\sigma|a+b} x \boxplus y'}$$

Recall that

$$\frac{}{X \xrightarrow{\sigma|\sigma(X)} X}$$

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The buffer operator breaks the soundness

Consider the following rules:

$$\frac{}{a.x \xrightarrow{\sigma | a} \sigma \rhd x} \qquad \frac{x \xrightarrow{\hat{\sigma} | a} x'}{\hat{\sigma} \rhd x \xrightarrow{\sigma | a} \sigma \rhd x}$$

Let $\hat{\sigma}$ be s.t. $\hat{\sigma}(X) = a$ then

$$a.X \xrightarrow{\sigma | a} \sigma \rhd X \qquad \hat{\sigma} \rhd X \xrightarrow{\sigma | \hat{\sigma}(X)} \sigma \rhd X$$

$\rho(X) = b.X$, then

- $\rho(a.X) = a.b.X$
- $\rho(\hat{\sigma} \rhd X) = \hat{\sigma} \rhd (b.X)$

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Second result

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If the Mealy machine specification does not use
the buffer operator ($\triangleright$),
the up-to substitution is compatible

Avoiding the buffer operator

1. We need to add a family of prefix operators to the stream specification. For all $a, b \in A$

$$\frac{x \xrightarrow{b} x'}{a.x \xrightarrow{a} b.x'}$$

2. Avoid using the parameters of the operators in the target of the conclusion

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{x \boxplus y \xrightarrow{a+b} x \boxplus y'} \quad \rightsquigarrow \quad \frac{x \xrightarrow{\sigma|a} x' \quad y \xrightarrow{\sigma|b} y'}{x \boxplus y \xrightarrow{\sigma|a+b} a.x' \boxplus y'}$$

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Final comments

Contribution

- Bisimulation over open terms for streams that is complete, i.e. if $s \equiv t$ then $s \sim_{\mathcal{M}} t$.
- Restrictions to ensure that *"up-to substitution"* is compatible.

What's next

- Can we define conditions to ensure that for $t_1 \equiv t_2$ there is finite up-to *[substitution/equivalence/bisimilarity/context]* bisimulation?

Extending the results to other transition systems

- We can extend the idea of bisimulation over open terms to LTS.
  - We need a set of equations to manipulate open terms. *"Turning SOS Rules into Equations"*, Aceto, Bloom, Vaandrager.
- Considering probabilities.
  - *"Axiomatizing Bisimulation Equivalences and Metrics from Probabilistic SOS Rules"*, D'Argenio, Gebler, Lee.