Bisimulation on Open Terms for Stream Systems

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Outline

1. Our contribution
   - Bisimulation over open terms.
   - Bisimulation up-to substitutions
2. Formalization of the results
   - Coalgebras
   - Stream GSOS Specifications
   - Mealy machines over open terms

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Operations over streams can be defined by means of stream GSOS specification

$$\langle \Sigma, A, R \rangle$$

- Σ a set of operations.
- A the set of actions that can be executed by the streams.
- R a set of rules, defining the semantic of the operators.

A well-known result:

Bisimilarity is a congruence.

Example (1/2)

- $\Sigma = \{\mathbf{a} \mid a \in A\} \cup \{\text{alt}\}$.
- A a set of actions.
- R a set of rules containing:

$$\frac{}{\mathbf{a} \xrightarrow{a} \mathbf{a}} \forall a \in A \qquad \frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\text{alt}(x, y) \xrightarrow{a} \text{alt}(y', x')} \forall a, b \in A$$

Example (2/2)

$$\frac{}{\mathbf{a} \xrightarrow{a} \mathbf{a}} \; \forall a \in A \qquad \frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\mathrm{alt}(x, y) \xrightarrow{a} \mathrm{alt}(y', x')} \; \forall a, b \in A$$

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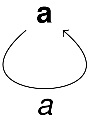

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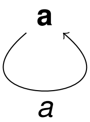

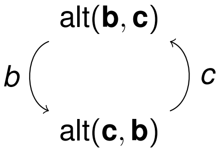

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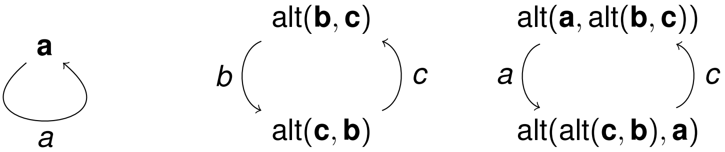

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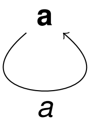

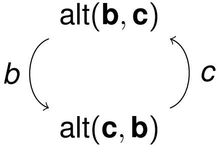

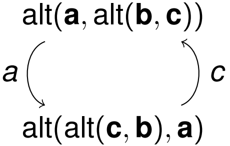

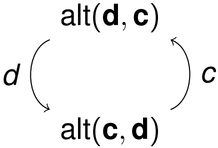

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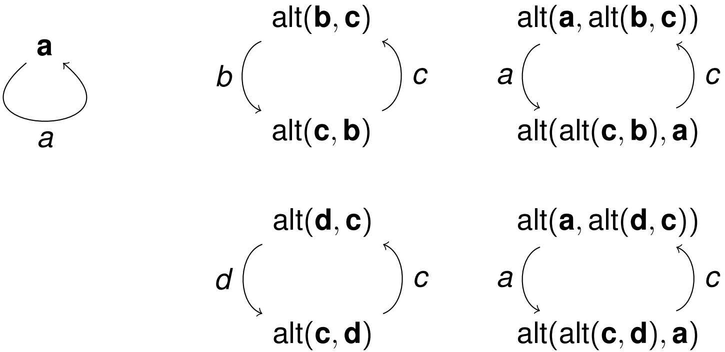
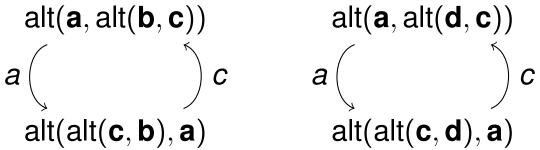


These systems are bisimilar



In general

$$\mathrm{alt}(\mathcal{X}, \mathrm{alt}(\mathcal{Y}, \mathcal{Z})) \quad \equiv \quad \mathrm{alt}(\mathcal{X}, \mathrm{alt}(\mathcal{W}, \mathcal{Z}))$$

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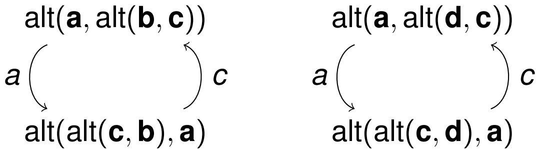

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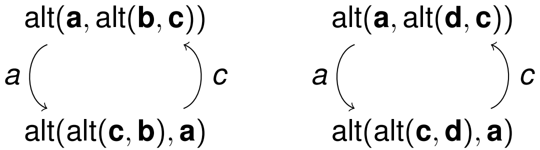

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- *Q*: How do we prove that the equation is sound?

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$\mathsf{alt}(\mathbf{a}, \mathsf{alt}(\mathbf{b}, \mathbf{c}))$ $\xrightarrow{a}$ $\mathsf{alt}(\mathsf{alt}(\mathbf{c}, \mathbf{b}), \mathbf{a})$ $\xrightarrow{c}$ $\mathsf{alt}(\mathbf{a}, \mathsf{alt}(\mathbf{b}, \mathbf{c}))$

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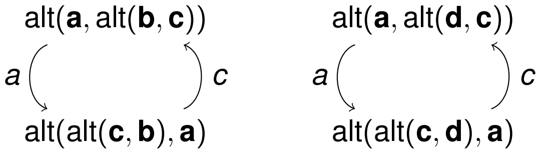
- *Q*: How do we prove that the equation is sound?
- *A*: For all possible closed instantiation of $\mathcal{X}$,$\mathcal{Y}$, $\mathcal{Z}$ check that both terms are bisimilar.

These systems are bisimilar



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- *Q*: How do we prove that the equation is sound?

- A': We define a transition systems over open terms based on the stream system. The equation is sound if both sides of the equation are bisimilar in the new model.

Transition systems over open terms - The intuition

$$\mathrm{alt}(\mathcal{X}, t) \text{ with } \mathcal{X} \in \mathcal{V}, t \in T_\Sigma \mathcal{V}$$

Recall

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\mathrm{alt}(x, y) \xrightarrow{a} \mathrm{alt}(y', x')} \; \forall a, b \in A$$

Transition systems (TS) over open terms

- Let $\varsigma : \mathcal{V} \to A$ be a function that defines the output that can be executed by a variable.
- We consider *Mealy machine* $(T_\Sigma \mathcal{V}, \alpha)$ with
  - inputs in $A^{\mathcal{V}}$ and
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Specification of TS over open terms

- New rules describing the behavior of the variables. For all $\varsigma : \mathcal{V} \to A$,

$$\frac{}{\mathcal{X} \xrightarrow{\varsigma|\varsigma(\mathcal{X})} \mathcal{X}}$$

- The original rules are lifted to rules for Mealy machines:

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\mathrm{alt}(x,y) \xrightarrow{a} \mathrm{alt}(y',x')} \quad \rightsquigarrow \quad \frac{x \xrightarrow{\varsigma|a} x' \quad y \xrightarrow{\varsigma|b} y'}{\mathrm{alt}(x,y) \xrightarrow{\varsigma|a} \mathrm{alt}(y',x')}$$

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Example:

Taking into account the rules

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we get for all $\varsigma, \varsigma' : \mathcal{V} \to A$

$$\mathrm{alt}(\mathcal{Y},\mathcal{Z}) \xrightarrow{\varsigma|\varsigma(\mathcal{Y})} \mathrm{alt}(\mathcal{Z},\mathcal{Y}) \xrightarrow{\varsigma'|\varsigma'(\mathcal{Z})} \mathrm{alt}(\mathcal{Y},\mathcal{Z})$$

$$\mathrm{alt}(\mathcal{X},\mathrm{alt}(\mathcal{Y},\mathcal{Z})) \xrightarrow{\varsigma|\varsigma(\mathcal{X})} \mathrm{alt}(\mathrm{alt}(\mathcal{Z},\mathcal{Y}),\mathcal{X}) \xrightarrow{\varsigma'|\varsigma'(\mathcal{Z})} \mathrm{alt}(\mathcal{X},\mathrm{alt}(\mathcal{Y},\mathcal{Z}))$$

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Soundness of the lifting

Later, we will prove something like: For all $t_1, t_2 \in T_\Sigma \mathcal{V}$:

$$t_1 \sim_{\mathcal{M}} t_2 \text{ iff } t_1 \equiv t_2$$

A stream calculus over $\mathbb{R}$

$$\frac{}{n \xrightarrow{n} 0} \qquad \frac{x \xrightarrow{m} x'}{n.x \xrightarrow{n} m.x'} \qquad \frac{x \xrightarrow{n} x' \quad y \xrightarrow{m} y'}{x \oplus y \xrightarrow{n+m} x' \oplus y'}$$

$$\frac{x \xrightarrow{n} x' \quad y \xrightarrow{m} y'}{x \otimes y \xrightarrow{n \times m} (n \otimes y') \oplus (x' \otimes m.y')}$$

- $2.3 \xrightarrow{2} 3 \xrightarrow{3} 0 \xrightarrow{0} \dots$
- $2 \oplus 3.4 \xrightarrow{5} 0 \oplus 4 \xrightarrow{4} 0 \oplus 0 \xrightarrow{0} \dots$
- $2 \otimes 3.4 \xrightarrow{6} (2 \otimes 4) \oplus (0 \otimes 3.4) \xrightarrow{8} \dots$

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We work with monadic GSOS specification!

Bisimulation up-to substitutions - Motivation

- Consider the unary operators f and g defined by:

$$\frac{x \xrightarrow{a} x'}{f(x) \xrightarrow{a} f(x' \oplus x')} \qquad \frac{x \xrightarrow{a} x'}{g(x) \xrightarrow{a} g(x' \oplus x')}$$

- For all closed terms t, $f(t) \sim g(t)$: if $t = t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \ldots$ then

$$f(t_0) \xrightarrow{a_0} f(t_1 \oplus t_1) \xrightarrow{a_1 + a_1} f((t_2 \oplus t_2) \oplus (t_2 \oplus t_2)) \ldots$$
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- Any bisimulation R relating the pair is infinite.

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- $f(\mathcal{X}) \sim g(\mathcal{X})$:

$$f(\mathcal{X}) \xrightarrow{\varsigma_0|\varsigma_0(\mathcal{X})} f(\mathcal{X} \oplus \mathcal{X}) \xrightarrow{\varsigma_1|\varsigma_1(\mathcal{X})+\varsigma_1(\mathcal{X})} f((\mathcal{X} \oplus \mathcal{X}) \oplus (\mathcal{X} \oplus \mathcal{X}))\dots$$
$$g(\mathcal{X}) \xrightarrow{\varsigma_0|\varsigma_0(\mathcal{X})} g(\mathcal{X} \oplus \mathcal{X}) \xrightarrow{\varsigma_1|\varsigma_1(\mathcal{X})+\varsigma_1(\mathcal{X})} g((\mathcal{X} \oplus \mathcal{X}) \oplus (\mathcal{X} \oplus \mathcal{X}))\dots$$

- Any bisimulation R relating the pair is infinite.

Up-to substitutions

Definition

A relation $R \subseteq T\mathcal{V} \times T\mathcal{V}$ is a bisimulation up to substitutions if $t_1 \; R \; t_2$ implies

- if $t_1 \xrightarrow{\varsigma | a} t_1'$ then $t_2 \xrightarrow{\varsigma | a} t_2'$ and $t_1' \; R_{\forall\rho} \; t_2'$.

where

$$R_{\forall\rho} := \{(\rho(t_1), \rho(t_2)) \mid t_1 \; R \; t_2, \rho : \mathcal{V} \to T\mathcal{V}\}$$

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Relation $R = \{(f(\mathcal{X}), g(\mathcal{X}))\}$ is a bisimulation up-to substitutions.

- $f(\mathcal{X}) \xrightarrow{\varsigma_0|1} f(\mathcal{X} \oplus \mathcal{X})$
- $g(\mathcal{X}) \xrightarrow{\varsigma_0|1} g(\mathcal{X} \oplus \mathcal{X})$
- $f(\mathcal{X} \oplus \mathcal{X})\rho(R)g(\mathcal{X} \oplus \mathcal{X})$ for $\rho(\mathcal{X}) = \mathcal{X} \oplus \mathcal{X}$.

Theorem

$(-)_{\forall\rho}$ *is compatible with b .*

Proof: using the theory in “Enhancements of the bisimulation proof method”, D. Pous and D. Sangiorgi.

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Example

$$\{(\mathcal{X} \otimes (\mathcal{Y} \oplus \mathcal{Y}), ((\mathcal{X} \otimes \mathcal{Y}) \oplus (\mathcal{X} \otimes \mathcal{Y}))\}$$

is a bisimulation up-to $\sim\mathcal{C}(\sim R_{\forall\rho}\sim)\sim$. Then

$$\mathcal{X} \otimes (\mathcal{Y} \oplus \mathcal{Y}) \equiv (\mathcal{X} \otimes \mathcal{Y}) \oplus (\mathcal{X} \otimes \mathcal{Y})$$

Outline

Coalgebras

Definition

Given a functor $F\colon set \to set$, an *F-coalgebra* is a pair $\langle X, d\rangle$, where

- X is a set (called the carrier) and
- $d\colon X \to FX$ is a function (called the structure).

Definition

A *stream system* over an alphabet A is a pair $\langle X, \langle o, d\rangle\rangle$ where

- X is a set of states and
- $\langle o, d\rangle\colon X \to A \times X$ is a transition function, which maps a state $x \in X$ to an *output value* $o(x) \in A$ and a *next state* $d(x) \in X$.

We write $x \xrightarrow{a} y$ whenever $o(x) = a$ and $d(x) = y$.

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Definition

A *Mealy machine* with inputs in a set B and outputs in a set A is a pair $\langle X, m\rangle$ where

- X is a set of states and
- $m\colon X \to (A \times X)^B$ is a transition function, which maps each $x \in X$ to another map $m(x) = \langle o_x, d_x\rangle\colon B \to A \times X$ s.t. for the input $b \in B$ in the state x, the output $o_x(b)$ is generated and the next state $d_x(b)$ is reached.

We write $x \xrightarrow{b|a} y$ whenever $o_x(b) = a$ and $d_x(b) = y$.

Final coalgebras

Definition

Given a functor $F\colon set \to set$, an *F-coalgebra morphism* from $d\colon X \to FX$ to $d'\colon Y \to FY$ is a map f such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ FX & \xrightarrow{\quad Ff \quad} & FY \end{array}$$

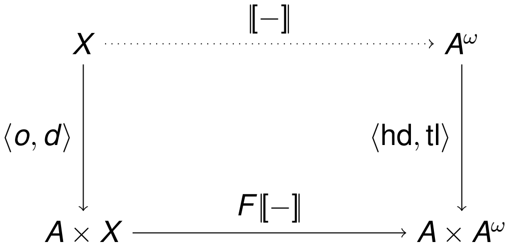
Definition

A coalgebra $\zeta\colon Z \to FZ$ is called *final* if

- for every F-coalgebra $d\colon X \to FX$ there is a unique morphism $[\![-]\!]\colon X \to Z$ from d to ζ.

We call $[\![-]\!]$ the *coinductive extension* of d.

Final coalgebra for stream systems



Final coalgebras for Mealy machines

$$
\begin{array}{ccc}
X & \xrightarrow{\ \llbracket - \rrbracket\ } & \Gamma(B^\omega, A^\omega) \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \zeta_M} \\
(A \times X)^B & \xrightarrow{\ F\llbracket - \rrbracket\ } & (A \times \Gamma(B^\omega, A^\omega))^B
\end{array}
$$

where $\Gamma(B^\omega, A^\omega) = \{f \colon B^\omega \to A^\omega \mid f \text{ is causal}\}$.

x and y are behavioural equivalent if $[\![x]\!] = [\![y]\!]$.

Under the condition that F preserves weak pullbacks, behavioural equivalence coincides with bisimilarity, i.e., $x \sim y$ iff $[\![x]\!] = [\![y]\!]$

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Definition

A *stream GSOS rule* r for a signature Σ and a set A is a rule

$$\frac{x_1 \xrightarrow{a_1} x_1' \quad \cdots \quad x_n \xrightarrow{a_n} x_n'}{f(x_1, \ldots, x_n) \xrightarrow{a} t} \tag{1}$$

where $f \in \Sigma$ with arity n, $x_1, \ldots, x_n, x_1', \ldots, x_n'$ are pairwise distinct variables, t is a term built over variables $\{x_1, \ldots, x_n, x_1', \ldots, x_n'\}$ and $a, a_1, \ldots, a_n \in A$. We say that r is triggered by $(a_1, \ldots, a_n) \in A^n$.

Definition

A *stream GSOS specification* is a tuple (Σ, A, R) where Σ is a signature, A is a set of actions and R is a set of stream GSOS rules for Σ and A s.t. for each $f \in \Sigma$ of arity n and each tuple $(a_1, \ldots, a_n) \in A^n$, there is only one rule $r \in R$ that is triggered by $(a_1, \ldots, a_n)$.

A stream GSOS specification:

from $\langle o, d \rangle : X \to A \times X$ to $\overline{\langle o, d \rangle} : TX \to A \times TX$.

$$\frac{}{n \xrightarrow{n} 0} \qquad \frac{x \xrightarrow{m} x'}{n.x \xrightarrow{n} m.x'} \qquad \frac{x \xrightarrow{n} x' \quad y \xrightarrow{m} y'}{x \oplus y \xrightarrow{n+m} x' \oplus y'}$$

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If $X = \emptyset$ and $\langle o, d\rangle = ! : \emptyset \to A \times \emptyset$ then

$$\overline{\langle o, d\rangle} : T\emptyset \to A \times T\emptyset$$

- $2.3 \xrightarrow{2} 3 \xrightarrow{3} 0 \xrightarrow{0} \ldots$
- $2 \oplus 3.4 \xrightarrow{5} 0 \oplus 4 \xrightarrow{4} 0 \oplus 0 \xrightarrow{0} \ldots$

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If $X = \mathbb{R}^\omega$ and $\langle o, d\rangle = \langle \mathrm{hd}, \mathrm{tl}\rangle : R^\omega \to \mathbb{R} \times R^\omega$ then

$$\overline{\langle o, d\rangle} : T\mathbb{R}^\omega \to A \times T\mathbb{R}^\omega$$

- $n.a_0 a_1 a_2 \ldots \xrightarrow{n} a_0 a_1 a_2 a_3 \ldots \xrightarrow{a_0} a_1 a_2 a_3 \xrightarrow{a_1} \ldots$
- $a_0 a_1 a_2 \ldots \oplus b_0 b_1 b_2 \ldots \xrightarrow{a_0+b_0} a_1 a_2 a_3 \ldots \oplus b_1 b_2 b_3 \ldots \xrightarrow{a_1+b_1} a_2 a_3 a_4 \ldots \oplus b_2 b_3 b_4 \ldots \xrightarrow{a_2+b_2} \ldots$

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If $X = \mathcal{V}$ and $\langle o, d\rangle = \ldots$ then

$$\overline{\langle o, d\rangle} : T\mathcal{V} \to A \times T\mathcal{V}$$

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If $X = \mathcal{V}$ and $\langle o, d\rangle = \ldots$ then

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Concrete and abstract semantics

$$
\begin{array}{ccc}
T\emptyset & \xrightarrow{\;[\![-]\!]_c\;} & A^\omega \\
{\scriptstyle \overline{\langle o, d\rangle}}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{hd}, \mathrm{tl}\rangle} \\
A \times T\emptyset & \xrightarrow{\;F[\![-]\!]_c\;} & A \times A^\omega
\end{array}
$$

$$
\begin{array}{ccc}
TA^\omega & \xrightarrow{\;[\![-]\!]_a\;} & A^\omega \\
{\scriptstyle \overline{\langle \mathrm{hd}, \mathrm{tl}\rangle}}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{hd}, \mathrm{tl}\rangle} \\
A \times TA^\omega & \xrightarrow{\;F[\![-]\!]_a\;} & A \times A^\omega
\end{array}
$$

From a categorical point of view, GSOS specifications are presented as distributive laws.

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Mealy machines over variables

c is a Mealy machines over variables s.t.

$$c\colon \mathcal{V} \to (A \times \mathcal{V})^{A^{\mathcal{V}}} \qquad c(\mathcal{X})(\varsigma) = (\varsigma(\mathcal{X}), \mathcal{X})\,. \qquad (2)$$

Graphically, c is the Mealy machine which has, for every $\varsigma\colon \mathcal{V} \to A$ and every variable $\mathcal{X} \in \mathcal{V}$, a transition

- $\mathcal{X} \xrightarrow{\varsigma_0|\varsigma_0(\mathcal{X})} \mathcal{X} \xrightarrow{\varsigma_1|\varsigma_1(\mathcal{X})} \mathcal{X} \xrightarrow{\varsigma_2|\varsigma_2(\mathcal{X})} \ldots$

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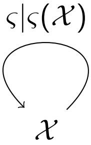

- $\mathcal{X} \xrightarrow{\varsigma_0|\varsigma_0(\mathcal{X})} \mathcal{X} \xrightarrow{\varsigma_1|\varsigma_1(\mathcal{X})} \mathcal{X} \xrightarrow{\varsigma_2|\varsigma_2(\mathcal{X})} \dots$

A Mealy GSOS specification:

from $m\colon X \to (A \times X)^{A^{\mathcal{V}}}$ to $\overline{m}\colon TX \to (A \times TX)^{A^{\mathcal{V}}}$.

$$\frac{}{n \xrightarrow{\varsigma|n} 0} \qquad \frac{x \xrightarrow{\varsigma|m} x'}{n.x \xrightarrow{\varsigma|n} m.x'} \qquad \frac{x \xrightarrow{\varsigma|n} x' \quad y \xrightarrow{\varsigma|m} y'}{x \oplus y \xrightarrow{\varsigma|n+m} x' \oplus y'}$$

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These rules induces a *pointwise extension* of the operators induced by the original rules. See:

"Pointwise extensions of GSOS-defined operations".
H. H. Hansen and B. Klin.

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If $X = \mathcal{V}$ and $m = c\colon \mathcal{V} \to (A \times \mathcal{V})^{A^{\mathcal{V}}}$ then

$$\overline{m}\colon T\mathcal{V} \to (A \times T\mathcal{V})^{A^{\mathcal{V}}}$$

- $n \xrightarrow{\varsigma_0|n} 0 \xrightarrow{\varsigma_1|n} 0 \xrightarrow{\varsigma_2|n} \ldots$
- $\mathcal{X} \xrightarrow{\varsigma_0|\varsigma_0(\mathcal{X})} \mathcal{X} \xrightarrow{\varsigma_1|\varsigma_1(\mathcal{X})} \mathcal{X} \xrightarrow{\varsigma_2|\varsigma_2(\mathcal{X})} \ldots$
- $\mathcal{X} \oplus \mathcal{Y} \xrightarrow{\varsigma_0|\varsigma_0(\mathcal{X})+\varsigma_0(\mathcal{Y})} \mathcal{X} \oplus \mathcal{Y} \xrightarrow{\varsigma_1|\varsigma_1(\mathcal{X})+\varsigma_1(\mathcal{X})} \mathcal{X} \oplus \mathcal{Y} \xrightarrow{\varsigma_2|\varsigma_2(\mathcal{X})+\varsigma_2(\mathcal{Y})} \ldots$

Open semantics

$$\begin{array}{ccc}
T\mathcal{V} & \xrightarrow{\llbracket - \rrbracket} & \Gamma(A^{\mathcal{V}^\omega}, A^\omega) \\
\downarrow{\scriptstyle \overline{m}} & & \downarrow{\scriptstyle \zeta_M} \\
(A \times T\mathcal{V})^{A^\mathcal{V}} & \xrightarrow{F\llbracket - \rrbracket} & (A \times \Gamma(A^{\mathcal{V}^\omega}, A^\omega))^{A^\mathcal{V}}
\end{array}$$

Using the fact that $A^{\mathcal{V}^\omega} \simeq A^{\omega^\mathcal{V}}$, we get

$$\begin{array}{ccc}
T\mathcal{V} & \xrightarrow{\llbracket - \rrbracket_o} & \Gamma(A^{\omega^\mathcal{V}}, A^\omega) \\
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(A \times T\mathcal{V})^{A^\mathcal{V}} & \xrightarrow{F\llbracket - \rrbracket_o} & (A \times \Gamma(A^{\omega^\mathcal{V}}, A^\omega))^{A^\mathcal{V}}
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Example of $[\![-]\!]_o$

- Let $\psi : \mathcal{V} \to A^\omega$ be such that

$$\psi(\mathcal{X}) = a_0 a_1 a_2 a_3 \ldots \qquad \psi(\mathcal{Y}) = b_0 b_1 b_2 b_3 \ldots$$

- then $[\![\mathcal{X} \oplus \mathcal{Y}]\!]_o \in \Gamma({A^\omega}^{\mathcal{V}}, A^\omega)$

$$[\![\mathcal{X} \oplus \mathcal{Y}]\!]_o(\psi) = (a_0 + b_0)(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \ldots \in A^\omega$$

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Proposition

Let $[\![-]\!]_a$ and $[\![-]\!]_o$ be the abstract and open semantics respectively of a monadic stream GSOS specification λ. Then for any $t \in T\mathcal{V}$ and any $\psi : \mathcal{V} \to A^\omega$:

$$[\![t]\!]_o(\psi) = [\![(T\psi)(t)]\!]_a\,.$$

Theorem

For all $t_1, t_2 \in T\mathcal{V}$,

$$[\![t_1]\!]_o = [\![t_2]\!]_o \textit{ iff } [\![T\psi(t_1)]\!]_a = [\![T\psi(t_2)]\!]_a \textit{ for all } \psi\colon \mathcal{V} \to A^\omega$$

Theorem

Suppose $\lambda \colon \Sigma(A \times -) \Rightarrow A \times T\Sigma$ *is a monadic stream GSOS specification which contains, for each* $a \in A$*, the prefix operator* $a.-$ *in monadic GSOS format. Further, assume* $T\emptyset$ *is non-empty.*

Let $[\![-]\!]_c$ *and* $[\![-]\!]_o$ *be the closed and open semantics respectively of* λ*. Then for all* $t_1, t_2 \in T\mathcal{V}$*:*

$$[\![t_1]\!]_o = [\![t_2]\!]_o \text{ iff } [\![\phi^\dagger(t_1)]\!]_c = [\![\phi^\dagger(t_2)]\!]_c \text{ for all } \phi \colon \mathcal{V} \to T\emptyset$$

Some remarks

- We also show that arbitrary GSOS specification can be translated to a monadic one.
- “Completeness”: can we ensure there is always a finite bisimulation up-to relating two bisimilar open terms?
- GSOS does not support lookahead, then we cannot introduce operatos as even and odd.
- Extension to other formats: yes, but in some cases we lose some up-to techniques.
- Extension to other models: LTS.