

Mechanization of Separation in Generic Extensions

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Abstract—We mechanize, in the proof assistant Isabelle, a proof of the axiom-scheme of Separation in generic extensions of models of set theory by using the fundamental theorems of forcing. We also formalize the satisfaction of the axioms of Extensionality, Foundation, Union, and Powerset. The axiom of Infinity is likewise treated, under additional assumptions on the ground model. In order to achieve these goals, we extended Paulson’s library on constructibility with renaming of variables for internalized formulas, improved results on definitions by recursion on well-founded relations, and sharpened hypotheses in his development of relativization and absoluteness.

I. INTRODUCTION

Zermelo-Fraenkel Set Theory (ZF) has a prominent place among formal theories. The reason for this is that it formalizes many intuitive properties of the notion of set. As such, it can be used as a foundation for mathematics and thus it has been thoroughly studied. Considering the current trend of mechanization of mathematics [1], it seems natural to ask for a mechanization of the most salient results of Set Theory.

The results we are interested in originally arose in connection to *relative consistency* proofs in set theory; that is, showing that if ZF is consistent, the addition of a new axiom A won’t make the system inconsistent; this is as much as we can expect to obtain, since Gödel’s Incompleteness Theorems precludes a formal proof of the consistency of set theory in ZF , unless the latter is indeed inconsistent. There are statements A which are *undecided* by ZF , in the sense that both A and $\neg A$ are consistent relative to ZF ; perhaps the most prominent is the *Continuum Hypothesis*, which led to the development of powerful techniques for independence proofs. First, Gödel inaugurated the theory of *inner models* by introducing his model L of the *Axiom of Constructibility* [11] and proved the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with ZF . More than twenty years later, Paul J. Cohen [5] devised the technique of *forcing*, which is the only known way of *extending* models of ZF ; this was used to prove the relative consistency of the negation of the Continuum Hypothesis.

In this work we address a substantial part of formalizing the proof that given a model M of ZF , any *generic extension* $M[G]$ obtained by forcing also satisfies ZF . As remarked by Kunen [17, p.250] “[...] in verifying that $M[G]$ is a model for set theory, the hardest axiom to verify is [Separation].” The most important achievement of this paper is the mechanization in the proof assistant Isabelle of a proof of the Axiom of

Separation in generic extensions by using the “fundamental” theorems of forcing. En route to this, we also formalized the satisfaction by $M[G]$ of Extensionality, Foundation, and Union. As a consequence of Separation we were able to formalize the proof of the Powerset Axiom; finally, the Axiom of Infinity was proved under extra assumptions. The theoretical support for this work has been the fine textbook by Kunen [17] and our development benefited from the remarkable work done by Lawrence Paulson [25] on the formalization of Gödel’s constructible universe in Isabelle.

The ultimate goal of our project is the formalization of the forcing techniques needed to show the independence of the Continuum Hypothesis. We think that this project constitutes an interesting test-case for the current technology of formalization of mathematics, in particular for the need of handling several layers of reasoning.

The *Formal Abstracts* project [13] proposes the formalization of complex pieces of mathematics by writing the statements of results and the material upon which they are based (definitions, propositions, lemmas), but omitting the proofs. In this work we partially adhere to this vision to delineate our formalization strategy: Since the proofs that the axioms hold in generic extensions are independent of the *proofs* of the fundamental theorems of forcing, we assumed the latter for the time being. Let us remark that those theorems depend on the definition of a function `forces` from formulas to formulas which is, by itself, quite demanding; the formalization of it and of the fundamental theorems of forcing comprises barely less than a half of our full project.

It might be a little surprising the lack of formalizations of forcing and generic extensions. As far as we know, the development of Quirin and Tabareau [26] in homotopy type theory for constructing generic extensions in a sheaf-theoretic setting is the unique mechanization of forcing. This contrast with the fruitful use of forcing techniques to extend the Curry-Howard isomorphism to classical axioms [19, 16]. Moreover, the combination of forcing with intuitionistic type theory [6, 7] gives rise both to positive results (an algorithm to obtain witnesses of the continuity of definable functionals [8]) and also negative (the independence of Markov’s principle [9]). In the same strand of forcing from the point of view of proof theory [2] are the conservative extensions of CoC with forcing conditions [15, 14].

In pursuing the proof of Separation on generic extensions

we extended Paulson’s library with: (i) renaming of variables for *internalized* formulas, which with little effort can be extended to substitutions; (ii) an improvement on definitions by recursion on well-founded relations; (iii) enhancements in the hierarchy of locales; and (iv) a variant of the principle of dependent choices and a version of Rasiowa-Sikorski, which ensures the existence of generic filters for countable and transitive models of *ZF*; the last item was already communicated in the first report [12].

We briefly describe the contents of each section. Section II contains the bare minimum requirements to understand the (meta)logics used in Isabelle. Next, an overview of the model theory of set theory is presented in Section III. There is an “internal” representation of first-order formulas as sets, implemented by Paulson; Section IV discusses syntactical transformations of the former, mainly permutation of variables. In Section V the generic extensions are succinctly reviewed and how the treatment of well founded recursion in Isabelle was enhanced. We take care of the “easy axioms” in Section VII; these are the ones that do not depend on the forcing theorems. We describe the latter in Section VIII. We adapted the work by Paulson to our needs, and this is described in Section VI. We present the proof of the Separation Axiom Scheme in Section IX, which follows closely its implementation, and some comments on the proof of the Powerset Axiom. A plan for future work and some immediate conclusions are offered in Section X.

II. ISABELLE

A. Logics

Isabelle [29, 21] provides a meta-language called *Pure* that consists of a fragment of higher order logic, where \Rightarrow is the function-space arrow. The meta-Boolean type is called *prop*. Meta-connectives \Longrightarrow and $\&\&\&$ fulfill the role of implication and conjunction, and the meta-binder \bigwedge corresponds to universal quantification.

On top of *Pure*, theories/object logics can be defined, with their own types, connectives and rules. Rules can be written using meta-implication: “*P*, *Q*, and *R* yield *S*” can be written

$$P \Longrightarrow Q \Longrightarrow R \Longrightarrow S$$

(as usual, \Longrightarrow associates to the right), and syntactic sugar is provided to curry the previous rule as follows:

$$\llbracket P; Q; R \rrbracket \Longrightarrow S.$$

One further example is given by induction on the natural numbers *nat*,

$$\llbracket P(0); (\bigwedge x. P(x) \Longrightarrow P(\text{succ}(x))) \rrbracket \Longrightarrow P(n),$$

where we are omitting the “typing” assumptions on *n* and *x*.

We work in the object theory *Isabelle/ZF*. Two types are defined in this theory: *o*, the object-Booleans, and *i*, sets. It must be observed that predicates (functions with arguments of type *i* with values in *o*) do not correspond to first-order formulas; in particular, those are not recursively constructed.

This will have concrete consequences in our strategy to approach the development. From the beginning, we had to resort to *internalized* formulas [25, Sect. 6], i.e. elements of type *i* that encode first-order formulas with a binary relation symbol, and the satisfaction predicate $\text{sats} : : "i \Rightarrow i \Rightarrow i \Rightarrow o"$ between a set model with an environment and an internalized formula (where the relation symbol is interpreted as membership). The set formula: *"i"* of internalized formulas is defined by recursion and hence it is possible to perform inductive arguments using them. In this sense, the object-logic level is further divided into *internal* and *external* sublevels.

The source code is written for the 2018 version of Isabelle and can be downloaded from

<https://cs.famaf.unc.edu.ar/~mpagano/forcing/>

(with minor modifications, it can be run in Isabelle2016-1). Most of it is presented in the (nowadays standard) declarative flavour called *Isar* [30], where intermediate statements in the course of a proof are explicitly stated, interspersed with automatic tactics handling more trivial steps. The goal is that the resulting text, a *proof document*, can be understood without the need of replaying it and viewing the proof state at each point.

B. Locales

Locales [3] provide a neat facility to encapsulate a context (fixed objects and assumptions on them) that is to be used in proving several theorems, as in usual mathematical practice. Furthermore, locales can be organized in hierarchies.

In this paper, locales have a further use. The *Fundamental Theorems of Forcing* we use talk about a specific map *forces* from formulas to formulas. The definition of *forces* is involved and we will not dwell on this now; but applications of those theorems do not require to know how it is defined. Therefore, we black-box it and pack everything in a locale called *forcing_thms* that assumes that there is such a map that satisfies the Fundamental Theorems.

III. AXIOMS AND MODELS OF SET THEORY

The axioms of Zermelo and Fraenkel (*ZF*) form a countably infinite list of first-order sentences in a language consisting of an only binary relation symbol \in . These include the axioms of Extensionality, Pairing, Union, Powerset, Foundation, Infinity, and two axiom-schemes collectively referred as (a) Axiom of Separation: *For every* A, a_1, \dots, a_n , *and a formula* $\psi(x_0, x_1, \dots, x_n)$, *there exists* $\{a \in A : \psi(a, a_1, \dots, a_n)\}$, and (b) Axiom of Replacement: *For every* A, a_1, \dots, a_n , *and a formula* $\psi(x, z, x_1, \dots, x_n)$, *if* $\forall x. \exists! z. \psi(x, z, x_1, \dots, x_n)$, *there exists* $\{b : \exists a \in A. \psi(a, b, a_1, \dots, a_n)\}$. An excellent introduction to the axioms and the motivation behind them can be found in Shoenfield [27].

A model of the theory *ZF* consists of a pair $\langle M, E \rangle$ where *M* is a set and *E* is a binary relation on *M* satisfying the axioms. Forcing is a technique to extend very special kind of models, where *M* is a countable transitive set (i.e., every element of *M* is a subset of *M*) and *E* is the membership

relation \in restricted to M . In this case we simply refer to M as a *countable transitive model* or *ctm*. The following result shows how to obtain ctms from weaker hypotheses.

Lemma 1. *If there exists a model $\langle N, E \rangle$ of ZF such that the relation E is well founded, then there exists a countable transitive one.*

Proof. (Sketch) The Löwenheim-Skolem Theorem ensures that there is an countable elementary submodel $\langle N', E \upharpoonright N' \rangle \preceq \langle N, E \rangle$ which must also be well founded; then the Mostowski collapsing function [17, Def. I.9.31] sends $\langle N', E \upharpoonright N' \rangle$ isomorphically to some $\langle M, \in \rangle$ with M transitive. \square

In this stage of our implementation, we chose a presentation of the ZF axioms that would be most compatible with the development by Paulson. For instance, the predicate `upair_ax :: "(i=>o)=>o"` takes a “class” (unary predicate) C as an argument and states that C satisfies the Pairing Axiom.

`upair_ax(C) == $\forall x[C]. \forall y[C]. \exists z[C]. \text{upair}(C, x, y, z)$`

Here, $\forall x[C].\varphi$ stands for $\forall x.C(x) \rightarrow \varphi$, *relative* quantification. All of the development of relativization by Paulson is written for a class model, so we set up a locale fixing a set M and using the class `##M := $\lambda x. x \in M$` as the argument.

```

locale M_ZF =
  fixes M
  assumes
    upair_ax:      "upair_ax(##M)"
  and ...
  and separation_ax:
    "[  $\varphi$  formula ; arity( $\varphi$ )=1  $\vee$  arity( $\varphi$ )=2 ]
     $\implies$ 
    ( $\forall a \in M. \text{separation}(\##M, \lambda x. \text{sats}(M, \varphi, [x, a]))$ )"
  and replacement_ax:
    "[  $\varphi$  formula; arity( $\varphi$ )=2  $\vee$  arity( $\varphi$ )=3 ]
     $\implies$ 
    ( $\forall a \in M. \text{strong\_replacement}(\##M, \lambda x y. \text{sats}(M, \varphi, [x, y, a]))$ )"

```

The rest of the axioms are also included. We single out Separation and Replacement: These are written for formulas with at most one extra parameter (meaning $n \leq 1$ in the formula ψ of the axiom-schemes above). Thanks to Pairing, these versions are equivalent to the usual formulations. We are only able to prove that the generic extension satisfies Separation for any particular number of parameters, but not in general. This is a consequence that induction on terms of type `o` is not available.

It is also possible define a predicate that states that a set satisfies a (possibly infinite) set of formulas, and then to state that “ M satisfies ZF” in a standard way. With the aforementioned restriction on parameters, it can be shown that this statement is equivalent to the set of assumptions of the locale `M_ZF`.

IV. RENAMING

In the course of our work we need to reason about renaming of formulas and its effect on their satisfiability. Internalized formulas are implemented using de Bruijn indices for variables and the arity of a formula φ gives the least natural number containing all the free variables in φ . Following Fiore et al. [10], one can understand the arity of a formula as the context of the free variables; notice that the arity of $\forall\varphi$ is the predecessor of the arity of φ . Renamings are, consequently, mappings between finite sets; since we can think of `succ(n)` as the coproduct $1+n = \{0\} \cup \{1, \dots, n\}$, then given a renaming $f: n \rightarrow m$, the unique morphism `id1 + f: 1+n \rightarrow 1+m` is used to rename free variables in a quantified formula.

Definition 2 (Renaming). Let φ be a formula of arity n and let $f: n \rightarrow m$, the renaming of φ by f , denoted $(\varphi)[f]$, is defined by recursion on φ :

$$\begin{aligned}
 (i \in j)[f] &= f i \in f j \\
 (i = j)[f] &= f i = f j \\
 (\neg\varphi)[f] &= \neg(\varphi)[f] \\
 (\varphi \wedge \psi)[f] &= (\varphi)[f] \wedge (\psi)[f] \\
 (\forall\varphi)[f] &= \forall(\varphi)[\text{id}_1 + f]
 \end{aligned}$$

As usual, if M is a set, a_0, \dots, a_{n-1} are elements of M , and φ is a formula of arity n , we write

$$M, [a_0, \dots, a_{n-1}] \models \varphi$$

to denote that φ is satisfied by M when i is interpreted as a_i ($i = 0, \dots, n-1$). We call the list $[a_0, \dots, a_{n-1}]$ the *environment*.

The action of renaming on environments re-indexes the variables. An easy proof connects satisfaction with renamings.

Lemma 3. *Let φ be a formula of arity n , $f: n \rightarrow m$ be a renaming, and let $\rho = [a_1, \dots, a_n]$ and $\rho' = [b_1, \dots, b_m]$ be environments of length n and m , respectively. If for all $i \in n$, $a_i = b_j$ where $j = f i$, then $M, \rho \models \varphi$ is equivalent to $M, \rho' \models (\varphi)[f]$.*

An important resource in Isabelle/ZF is the facility for defining inductive sets [24, 23] together with a principle for defining functions by structural recursion. Internalized formulas are a prime example of this, so we define a function `ren` that associates to each formula an internalized function that can be later applied to suitable arguments. Notice that Paulson used `Nand` because it is more economical.

```
consts ren :: "i=>i"
```

```
primrec
```

```

  "ren(Member(x,y)) =
    ( $\lambda n \in \text{nat} . \lambda m \in \text{nat} . \lambda f \in n \rightarrow m. \text{Member}(f'x, f'y)$ )"

```

```

  "ren(Equal(x,y)) =
    ( $\lambda n \in \text{nat} . \lambda m \in \text{nat} . \lambda f \in n \rightarrow m. \text{Equal}(f'x, f'y)$ )"

```

```
"ren(Nand(p,q)) =
  (\ n \in nat . \ m \in nat. \ f \in n \to m.
    Nand (ren(p) 'n' m'f, ren(q) 'n' m'f))"
```

```
"ren(Forall(p)) =
  (\ n \in nat . \ m \in nat. \ f \in n \to m.
    Forall (ren(p) 'succ(n) 'succ(m) 'sum_id(n,f)))"
```

In the last equation, `sum_id` corresponds to the coproduct morphism $\text{id}_1 + f: 1 + n \rightarrow 1 + n$. Since the schema for recursively defined functions does not allow parameters, we are forced to return a function of three arguments (n, m, f) . This also exposes some inconveniences of working in the untyped realm of set theory; for example to use `ren` we will need to prove that the renaming is a function. Besides some auxiliary results (for example that the application of renaming to suitable arguments yields a formula), the main result corresponding to Lemma 3 is:

lemma `sats_iff_sats_ren` :

```
fixes \
assumes "\ \varphi \in formula"
shows "\ \bigwedge n m \rho \rho' f .
  [\ n \in nat ; m \in nat ; f \in n \to m ; arity(\varphi) \leq n ;
    \ \rho \in list(M) ; \ \rho' \in list(M) ;
    \ i . i < n \implies nth(i, \rho) = nth(f 'i, \rho') ] \implies
  sats(M, \varphi, \rho) \iff sats(M, ren(\varphi) 'n' m'f, \rho')"
```

All our uses of this lemma involve concrete renamings on small numbers, but we also tested it with more abstract ones for arbitrary numbers. All the renamings of the first kind follow the same pattern and, more importantly, share equal proofs. We would like to develop some ML tools in order to automatize this.

V. GENERIC EXTENSIONS

We will swiftly review some definitions in order to reach the concept of *generic extension*. As first preliminary definitions, a *forcing notion* $(\mathbb{P}, \leq, \mathbb{1})$ is simply a preorder with top, and a *filter* $G \subseteq \mathbb{P}$ is an increasing subset which is downwards compatible. Given a ctm M of ZF , a forcing notion in M , and a filter G , a new set $M[G]$ is defined. Each element $a \in M[G]$ is determined by its *name* \dot{a} of M . Actually, the structure of each \dot{a} is used to construct a . They are related by a map `val` that takes G as a parameter:

$$\text{val}(G, \dot{a}) = a.$$

Then the extension is defined by the image of the map `val` (G, \cdot):

$$M[G] := \{\text{val}(G, \tau) : \tau \in M\}.$$

Metatheoretically, it is straightforward to see that $M[G]$ is a transitive set that satisfies some axioms of ZF (see Section VII) and includes $M \cup \{G\}$. Nevertheless there is no a priori reason for $M[G]$ to satisfy either Separation, Powerset or Replacement. The original insight by Cohen was to define the notion of *genericity* for a filter $G \subseteq \mathbb{P}$ and to prove that

whenever G is generic, $M[G]$ will satisfy ZF . Remember that a filter is generic if it intersects all the dense sets in M ; in [12] we formalized the Rasiowa-Sikorski lemma which proves the existence of generic filters for ctm's.

The Separation Axiom is the first that requires the notion of genericity and the use of the forcing machinery, which we review in the Section VIII.

A. Recursion and values of names

The map `val` used in the definition of the generic extension is characterized by the recursive equation

$$\text{val}(G, \tau) = \{\text{val}(G, \sigma) : \exists p \in \mathbb{P}. \langle \sigma, p \rangle \in \tau \wedge p \in G\} \quad (1)$$

As is well-known, the principle of recursion on well-founded relations [17, p. 48] allows us to define a recursive function $F: A \rightarrow A$ by choosing a well-founded relation $R \subseteq A \times A$ and a functional $H: A \times (A \rightarrow A) \rightarrow A$ satisfying $F(a) = H(a, F \upharpoonright (R^{-1}(a)))$. Paulson [23] made this principle available in Isabelle/ZF via the the operator `wfrec`. The formalization of the corresponding functional `Hv` for `val` is straightforward:

definition

```
Hv :: "i \Rightarrow i \Rightarrow i \Rightarrow i" where
  "Hv(G, y, f) == {f 'x .. x \in domain(y), \exists p \in P.
  \langle x, p \rangle \in y \wedge p \in G}"
```

In the references [17, 28] `val` is applied only to *names*, that are certain elements of M characterized by a recursively defined predicate. The well-founded relation used to justify Equation (1) is

$$x \text{ ed } y \iff \exists p. \langle x, p \rangle \in y.$$

In order to use `wfrec` the relation should be expressed as a set, so in [12] we originally took the restriction of `ed` to the whole universe M ; i.e. $\text{ed} \cap M \times M$. Although this decision was adequate for that work, we now required more flexibility (for instance, in order to apply `val` to arguments that we can't assume that are in M , see Eq. (7) below).

The remedy is to restrict `ed` to the transitive closure of the actual parameter:

definition

```
val :: "i \Rightarrow i \Rightarrow i" where
  "val(G, \tau) == wfrec(edrel(eclose(\{\tau\})), \tau, Hv(G))"
```

In order to show that this definition satisfies (1) we had to supplement the existing recursion tools with a key, albeit intuitive, result stating that when computing the value of a recursive function on some argument a , one can restrict the relation to some ambient set if it includes a and all of its predecessors.

lemma

`wfrec_restr` :

```
assumes "relation(r)" "wf(r)"
shows "\ a \in A \implies (r^+)^- '\{a\}' \subseteq A \implies
  wfrec(r, a, H) = wfrec(r \cap A \times A, a, H)"
```

As a consequence, we are able to formalize Equation (1) as follows:

lemma *def_val*:

```
"val(G,x) = {val(G,t) .. t∈domain(x) ,
              ∃p∈P. <t,p>∈x ∧ p∈G }"
```

and the monotonicity of *val* follows automatically after a substitution.

lemma *val_mono*: " $x \subseteq y \implies \text{val}(G,x) \subseteq \text{val}(G,y)$ "
by (*subst* (1 2) *def_val*, *force*)

More interestingly we can give a neat equation for values of names defined by Separation, say $B = \{x \in A \times \mathbb{P}. Q(x)\}$, then

$$\text{val}(G,B) = \{\text{val}(G,t) : t \in A, \exists p \in \mathbb{P} \cap G. Q(\langle t,p \rangle)\} \quad (2)$$

We close our discussion of names and their values by making explicit the names for elements in M ; once more, we refer to [12] for our formalization. The definition of *check*(x) is a straightforward \in -recursion:

$$\text{check}(x) := \{\langle \text{check}(y), \mathbb{1} \rangle : y \in x\}$$

An easy \in -induction shows $\text{val}(G, \text{check}(x)) = x$. But to conclude $M \subseteq M[G]$ one also needs to have $\text{check}(x) \in M$; this result requires the internalization of recursively defined functions. This is also needed to prove $G \in M[G]$; let us define $\dot{G} = \{\langle \text{check}(p), p \rangle : p \in \mathbb{P}\}$, it is easy to prove $\text{val}(G, \dot{G}) = G$. Proving $\dot{G} \in M$ involves knowing $\text{check}(x) \in M$ and using one instance of Replacement.

Paulson proved absoluteness results for definitions by recursion and one of our next goals is to instantiate at $##M$ the appropriate locale M_{eclose} which is the last layer of a pile of locales. It will take us some time to prove that any ctm of ZF satisfies the assumptions involved in those locales; as we mentioned, Paulson's work is mostly done *externally*, i.e. the assumptions are instances of Separation and Replacement where the predicates and functions are Isabelle functions of type $i \Rightarrow i$ and $i \Rightarrow o$, respectively. In contrast, we assume that M is a model of ZF , therefore to deduce that M satisfies a Separation instance, we have to define an internalized formula whose satisfaction is equivalent to the external predicate (cf. the interface described in Section III and also the concrete example given in the proof of Union below).

In the meantime, we declare a locale $M_{\text{extra_assms}}$ assembling both assumptions (M being closed under *check* and the instance of Replacement); in this paper we explicitly mention where we use them.

VI. HACKING OF ZF-Constructible

In [25], Paulson presented his formalization of the relative consistency of the Axiom of Choice. This development is included inside the Isabelle distribution under the session `ZF-Constructible`. The main technical devices, invented by Gödel for this purpose, are *relativization* and *absoluteness*. In a nutshell, to relativize a formula φ to a class C , it is enough to restrict its quantifiers to C . The example of `upair_ax`

in Section III, the relativized version of the Pairing Axiom, is extracted from `Relative`, one of the core theories of `ZF-Constructible`. On the other hand, φ is *absolute* for C if it is equivalent to its relativization, meaning that the statement made by φ coincides with what C “believes” φ is saying. Paulson shows that under certain hypotheses on a class M (condensed in the locale `M_trivial`), a plethora of absoluteness and closure results can be proved about M .

The development of forcing, and the study of ctms in general, takes absoluteness as a starting point. We were not able to work with `ZF-Constructible` right out-of-the-box. The main reason is that we can't expect to state the “class version” of Replacement for a *set* M by using first-order formulas, since predicates $P: "i \Rightarrow o"$ can't be proved to be only the definable ones. Therefore, we had to make some modifications in several locales to make the results available as tools for the present and future developments.

The most notable changes, located in the theories `Relative` and `WF_absolute`, are the following:

- 1) The locale `M_trivial` does not assume that the underlying class M satisfies the relative Axiom of replacement. As a consequence, the lemma `strong_replacementI` is no longer valid and was commented out.
- 2) Originally the Powerset Axiom was assumed by the locale `M_trivial`, we moved this requirement to `M_basic`.
- 3) We replaced the need that the set of natural numbers is in M by the milder hypothesis that $M(0)$. Actually, most results should follow by only assuming that M is non-empty.
- 4) We moved the requirement $M(\text{nat})$ to the locale `M_trancl`, where it is needed for the first time. Some results, for instance `rtran_closure_mem_iff` and `iterates_imp_wfrec_replacement` had to be moved inside that locale.

Because of these changes, some theory files from the `ZF-Constructible` session have been included among ours.

The proof, for instance, that the constructible universe L satisfies the modified locale `M_trivial` holds with minor modifications. Nevertheless, in order to have a neater presentation, we have stripped off several sections concerning L from the theories `L_axioms` and `Internalize`, and we merged them to form the new file `Internalizations`.

VII. EXTENSIONALITY, FOUNDATION, UNION, INFINITY

In our first presentation of this project [12], we proved that $M[G]$ satisfies Pairing; now we have redone this proof in Isar. It is straightforward to show that the generic extension $M[G]$ satisfies extensionality and foundation. Showing that it is closed under Union depends on G being a filter. Infinity is also easy, but it depends in one further assumption.

For Extensionality in $M[G]$, the assumption $\forall w[M[G]]. w \in x \leftrightarrow w \in y$ yields $\forall w.w \in x \leftrightarrow w \in y$ by transitivity of $M[G]$. Therefore, by (ambient) Extensionality we conclude $x = y$.

Foundation for $M[G]$ does not depend on $M[G]$ being transitive: in this case we take $x \in M[G]$ and prove, relativized

to $M[G]$, that there is an \in -minimal element in x . Instantiating the global Foundation Axiom for $x \cap M[G]$ we get a minimal y , so it is still minimal when considered relative to $M[G]$.

It is noteworthy that the proofs in the Isar dialect of Isabelle strictly follow the argumentation of the two previous paragraphs.

The Union Axiom asserts that if x is a set, then there exists another set (the union of x) containing all the elements in each element of x . The relativized version of Union asks to give a name π_a for each $a \in M[G]$ and proving $val(G, \pi_a) = \bigcup a$. Let τ be the name for a , i.e. $a = val(G, \tau)$; Kunen [17] gives π_a in terms of τ :

$$\pi_a = \{ \langle \theta, p \rangle : \exists \langle \sigma, q \rangle \in \tau. \exists r. \langle \theta, r \rangle \in \sigma \wedge p \leq r \wedge p \leq q \}$$

Our formal definition is slightly different in order to ease the proof of $\pi_a \in M$; as it is defined using Separation, so one needs to define the domain of separation and also internalize the predicate as a formula `union_name_fm`. Instead of working directly with the internalized formula, we define a predicate `Union_name_body` and prove the equivalence between

$$sats(M, union_name_fm, [P, leq, \tau, x])$$

and `Union_Name_body(P, leq, \tau, x)`. The definition of π_a in our formalization is:

definition `Union_name` :: "i \Rightarrow i" **where**

```
"Union_name(\tau) ==
  {u \in domain(\bigcup (domain(\tau))) \times P .
   Union_name_body(P, leq, \tau, u)}"
```

Once we know $\pi_a \in M$, the equation $val(G, \pi_a) = \bigcup a$ is proved by showing the mutual inclusion; in both cases one uses that G is a filter.

lemma `Union_MG_Eq` :

```
assumes "a \in M[G]" and "a = val(G, \tau)" and
  "filter(G)" and "\tau \in M"
shows "\bigcup a = val(G, Union_name(\tau))"
```

Since Union is absolute for any transitive class we may conclude that $M[G]$ is closed under Union:

lemma `union_in_MG` :

```
assumes "filter(G)"
shows "Union_ax(##M[G])"
```

The proof of Infinity for $M[G]$ takes advantage of some absoluteness results proved in the locale `M_trivial`; this proof is easy because we work in the context of the locale `M_extra_assms` which states the assumption $check(x) \in M$ whenever $x \in M$. Since we have already proved that $M[G]$ is transitive, $\emptyset \in M[G]$ assuming G being generic, and also that it satisfies Pairing and Union, we can instantiate `M_trivial`:

sublocale `G_generic` \subseteq `M_trivial`"##M[G]"

We assume that M satisfies Infinity, i.e., that Infinity relativized to M holds; therefore we obtain $I \in M$ such that $\emptyset \in I$ and, $x \in I$ implies $succ(x) \in I$ by absoluteness of empty

and successor for M . Using the assumption that M is closed under `check`, we deduce $val(G, check(I)) = I \in M[G]$. Now we can use absoluteness of emptiness and successor, this time for $M[G]$, to conclude that $M[G]$ satisfies Infinity.

VIII. FORCING

For the most part, we follow Kunen [17]. As an alternative, introductory resource, the interested reader can check [4]; the book [28] contains a thorough treatment minimizing the technicalities.

Given a ctm M , and an M -generic filter $G \subseteq \mathbb{P}$, the Forcing Theorems relate satisfaction of a formula φ in the generic extension $M[G]$ to the satisfaction of another formula $forces(\varphi)$ in M . The map `forces` is defined by recursion on the structure of φ . It is to be noted that the base case (viz., for atomic φ) contains all the complexity; the case for connectives and quantifiers is then straightforward. In order to state the properties of this map in sufficient generality to prove that $M[G]$ satisfies *ZF*, we work with internalized formulas, because it is not possible to carry inductive arguments over \mathcal{O} .

We will now make more precise the properties of the map `forces` and how it relates satisfaction in M to that in $M[G]$. Actually, if the formula φ has n free variables, $forces(\varphi)$ will have $n+4$ free variables, where the first four account for the forcing notion and a particular element of it.

We write $\varphi(x_0, \dots, x_n)$ to indicate that the free variables of φ are among $\{x_0, \dots, x_n\}$. In the case of a formula of the form $forces(\varphi)$, we will make an abuse of notation and indicate the variables inside the argument of `forces`. As an example, take the formula $\varphi := x_1 \in x_0$. Then

$$M, [a, b] \models x_1 \in x_0$$

will hold whenever $b \in a$; and instead of writing $forces(\varphi)$ we will write $forces(x_5 \in x_4)$, as in

$$M, [\mathbb{P}, \leq, \mathbb{1}, p, \tau, \rho] \models forces(x_5 \in x_4).$$

If $\varphi = \varphi(x_0, \dots, x_n)$, the notation used by Kunen [17, 18] for $forces(\varphi)$ is

$$p \Vdash_{\mathbb{P}, \leq, \mathbb{1}}^* \varphi(x_0, \dots, x_n).$$

Here, the extra parameters are $\mathbb{P}, \leq, \mathbb{1}$, and $p \in \mathbb{P}$, and the first three are usually omitted. Afterwards, the *forcing relation* \Vdash can be obtained by interpreting \Vdash^* in a ctm M , for fixed $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in M$: $p \Vdash \varphi(\tau_0, \dots, \tau_n)$ holds if and only if

$$M, [\mathbb{P}, \leq, \mathbb{1}, p, \tau_0, \dots, \tau_n] \models forces(x_4, \dots, x_{n+4}). \quad (3)$$

A. The fundamental theorems

Modern treatments of the theory of forcing start by defining the forcing relation semantically and later it is proved that the characterization given by (3) indeed holds, and hence the forcing relation is *definable*.

Then the definition of the forcing relation is stated as a

Lemma 4 (Definition of Forcing). *Let M be a ctm of ZF, $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ a forcing notion in M , $p \in \mathbb{P}$, and $\varphi(x_0, \dots, x_n)$*

a formula in the language of set theory with all free variables displayed. Then the following are equivalent, for all $\tau_0, \dots, \tau_n \in M$:

- 1) $M, [\mathbb{P}, \leq, \mathbb{1}, p, \tau_0, \dots, \tau_n] \models \text{forces}(\varphi(x_4, \dots, x_{n+4}))$.
- 2) For all M -generic filters G such that $p \in G$, $M[G], [\text{val}(G, \tau_0), \dots, \text{val}(G, \tau_n)] \models \varphi(x_0, \dots, x_n)$.

The *Truth Lemma* states that the forcing relation indeed relates satisfaction in $M[G]$ to that in M .

Lemma 5 (Truth Lemma). *Assume the same hypothesis of Lemma 4. Then the following are equivalent, for all $\tau_0, \dots, \tau_n \in M$, and M -generic G :*

- 1) $M[G], [\text{val}(G, \tau_0), \dots, \text{val}(G, \tau_n)] \models \varphi(x_0, \dots, x_n)$.
- 2) There exists $p \in G$ such that $M, [\mathbb{P}, \leq, \mathbb{1}, p, \tau_0, \dots, \tau_n] \models \text{forces}(\varphi(x_4, \dots, x_{n+4}))$.

The previous two results combined are the ones usually called the *fundamental theorems*.

The following auxiliary results (adapted from [17, IV.2.43]) are also handy in forcing arguments.

Lemma 6 (Strengthening). *Assume the same hypothesis of Lemma 4. $M, [\mathbb{P}, \leq, \mathbb{1}, p, \dots] \models \text{forces}(\varphi)$ and $p_1 \leq p$ implies $M, [\mathbb{P}, \leq, \mathbb{1}, p_1, \dots] \models \text{forces}(\varphi)$.*

Lemma 7 (Density). *Assume the same hypothesis of Lemma 4. $M, [\mathbb{P}, \leq, \mathbb{1}, p, \dots] \models \text{forces}(\varphi)$ if and only if*

$$\{p_1 \in \mathbb{P} : M, [\mathbb{P}, \leq, \mathbb{1}, p_1, \dots] \models \text{forces}(\varphi)\}$$

is dense below p .

All these results are proved by recursion in formula.

The locale `forcing_thms` includes all these results as assumptions on the mapping `forces`, plus a typing condition and its effect on arities:

```

locale forcing_thms = forcing_data +
fixes forces :: "i  $\Rightarrow$  i"
assumes definition_of_forces:
  "p  $\in$  P  $\Rightarrow$   $\varphi \in$  formula  $\Rightarrow$  env  $\in$  list(M)  $\Rightarrow$ 
    sats(M, forces( $\varphi$ ), [P, leq, one, p] @ env)  $\longleftrightarrow$ 
    ( $\forall$  G. (M_generic(G)  $\wedge$  p  $\in$  G)  $\longrightarrow$ 
      sats(M[G],  $\varphi$ , map(val(G), env)))"
and definability[TC]: " $\varphi \in$  formula  $\Rightarrow$ 
  forces( $\varphi$ )  $\in$  formula"
and arity_forces: " $\varphi \in$  formula  $\Rightarrow$ 
  arity(forces( $\varphi$ )) = arity( $\varphi$ ) #+ 4"
and ...

```

The presentation of the Fundamental Theorems of Forcing in a locale can be regarded as a *formal abstract* as in the project envisioned by Hales [13], where statements of mathematical theorems proven in the literature are posed in a language that is both human- and computer-readable. The point is to take particular care so that, v.g., there are no missing hypotheses, so it is possible to take this statement as firm ground on which to start a formalization of a proof.

IX. SEPARATION AND POWERSSET

We proceed to describe in detail the main goal of this paper, the formalization of the proof of the Separation Axiom. Afterwards, we sketch the implementation of the Powerset Axiom.

This proof of Separation can be found in the file `Separation_Axiom.thy`. The order chosen to implement the proof sought to minimize the cross-reference of facts; it is not entirely appropriate for a text version, so we depart from it in this presentation. Nevertheless, we will refer to each specific block of code by line number for ease of reference.

The key technical result is the following:

lemma `Collect_sats_in_MG` :

assumes

" $\pi \in M$ " " $\sigma \in M$ " " $\text{val}(G, \pi) = c$ "

" $\text{val}(G, \sigma) = w$ "

" $\varphi \in$ formula" " $\text{arity}(\varphi) \leq 2$ "

shows

" $\{x \in c. \text{sats}(M[G], \varphi, [x, w])\} \in M[G]$ "

From this, using absoluteness, we will be able to derive the φ -instance of Separation.

To show that

$$S := \{x \in c : M[G], [x, w] \models \varphi(x_0, x_1)\} \in M[G],$$

it is enough to provide a name $n \in M$ for this set.

The candidate name is

$$n := \{u \in \text{dom}(\pi) \times \mathbb{P} : M, [u, \mathbb{P}, \leq, \mathbb{1}, \sigma, \pi] \models \psi\} \quad (4)$$

where

$$\psi := \exists \theta p. x_0 = \langle \theta, p \rangle \wedge \text{forces}(\theta \in x_5 \wedge \varphi(\theta, x_4)).$$

The fact that $n \in M$ follows (lines 216–220 of the source file) by an application of a six-variable instance of Separation in M (lemma `six_sep_aux`). We note in passing that it is possible to abbreviate expressions in Isabelle by the use of **let** statements or **is** qualifiers, and metavariables (whose identifiers start with a question mark). In this way, the definition in (4) appears in the sources as letting `?n` to be that set (lines 208–211).

Almost a third part of the proof involves the syntactic handling of internalized formulas and permutation of variables. The more substantive portion concerns proving that actually $\text{val}(G, n) = S$.

Let's first focus into the predicate

$$M, [u, \mathbb{P}, \leq, \mathbb{1}, \sigma, \pi] \models \psi \quad (5)$$

defining n by separation. By definition of the satisfaction relation and absoluteness, we have (lines 92–98) that it is equivalent to the fact that there exist $\theta, p \in M$ with $u = \langle \theta, p \rangle$ and

$$M, [\mathbb{P}, \leq, \mathbb{1}, p, \theta, \sigma, \pi] \models \text{forces}(x_4 \in x_6 \wedge \varphi(x_4, x_5)).$$

This, in turn, is equivalent by the Definition of Forcing to: *For all M -generic filters F such that $p \in F$,*

$$M[F], [val(F, \theta), val(F, \sigma), val(F, \pi)] \models x_0 \in x_2 \wedge \varphi(x_0, x_1). \quad (6)$$

(lines 99–185). We can instantiate this statement with G and obtain (lines 186–206)

$$p \in G \rightarrow M[G], [val(G, \theta), w, c] \models x_0 \in x_2 \wedge \varphi(x_0, x_1).$$

Let $Q(\theta, p)$ be the last displayed statement. We have just seen that (5) implies

$$\exists \theta, p \in M. u = \langle \theta, p \rangle \wedge Q(\theta, p).$$

Hence (lines 207–212) n is included in

$$m := \{u \in \text{dom}(\pi) \times \mathbb{P} : \exists \theta, p \in M. u = \langle \theta, p \rangle \wedge Q(\theta, p)\}.$$

Since m is a name defined using Separation, we may use (2) to show (lines 221–274 of Separation_Axiom)

$$val(G, m) = \{x \in c : M[G], [x, w, c] \models \varphi(x_0, x_1)\}. \quad (7)$$

The right-hand side is trivially equal to S , but as a consequence of the definition of separation_ax, the result contains an extra c in the environment.

Also, by monotonicity of val we obtain $val(G, n) \subseteq val(G, m)$ (lines 213–215). To complete the proof, it is therefore enough to show the other inclusion (starting at line 275). For this, let $x \in val(G, m) = S$ and then $x \in c$. Hence there exists $\langle \theta, q \rangle \in \pi$ such that $q \in G$ and $x = val(G, \theta)$.

On the other hand, since (line 297)

$$M[G], [val(G, \theta), val(G, \sigma), val(G, \pi)] \models x_0 \in x_2 \wedge \varphi(x_0, x_1),$$

by the Truth Lemma there must exist $r \in G$ such that

$$M, [\mathbb{P}, \leq, \mathbb{1}, r, \theta, \sigma, \pi] \models \text{forces}(x_4 \in x_6 \wedge \varphi(x_4, x_5)).$$

Since G is a filter, there is $p \in G$ such that $p \leq q, r$. By Strengthening, we have

$$M, [\mathbb{P}, \leq, \mathbb{1}, p, \theta, \sigma, \pi] \models \text{forces}(x_4 \in x_6 \wedge \varphi(x_4, x_5)),$$

which by the Definition of Forcing gives us (lines 315–318): *for all M -generic F , $p \in F$ implies*

$$M[F], [val(F, \theta), val(F, \sigma), val(F, \pi)] \models x_0 \in x_2 \wedge \varphi(x_0, x_1).$$

Note this is the same as (6). Hence, tracing the equivalence up to (5), we can show that $x = val(G, \theta) \in val(G, n)$ (lines 319–337), finishing the main lemma.

The last 20 lines of the theory show, using absoluteness, the two instances of Separation for $M[G]$:

theorem separation_in_MG:

assumes

" $\varphi \in \text{formula}$ " and

" $\text{arity}(\varphi) = 1 \vee \text{arity}(\varphi) = 2$ "

shows

" $\forall a \in (M[G]).$

$\text{separation}(\#\#M[G], \lambda x. \text{sats}(M[G], \varphi, [x, a]))$ "

We now turn to the Powerset Axiom. We followed the proof of [17, IV.2.27], to which we refer the reader for further details. Actually, the main technical result,

lemma Pow_inter_MG:

assumes

" $a \in M[G]$ "

shows

" $\text{Pow}(a) \cap M[G] \in M[G]$ "

keeps most of the structure of the printed proof; this “skeleton” of the argument takes around 120 (short) lines, where we tried to preserve the names of variables used in the textbook (with the occasional question mark that distinguishes meta-variables). There are approximately 30 more lines of bureaucracy in the proof of the last lemma.

Two more absoluteness lemmas concerning powersets were needed: These are refinements of results (powerset_Pow and powerset_imp_subset_Pow) located in the theory Relative where we weakened the assumption “ $y \in M$ ” ($M(y)$) to “ $y \subseteq M$ ” (second assumption below).

lemma (in M_trivial) powerset_subset_Pow:

assumes

"powerset(M, x, y)" " $\bigwedge z. z \in y \implies M(z)$ "

shows

" $y \subseteq \text{Pow}(x)$ "

lemma (in M_trivial) powerset_abs:

assumes

" $M(x)$ " " $\bigwedge z. z \in y \implies M(z)$ "

shows

"powerset(M, x, y) $\longleftrightarrow y = \{a \in \text{Pow}(x) . M(a)\}$ "

Of the rest of the theory file Powerset_Axiom.thy, a considerable fraction is taken by the proof of a closure property of the ctm M , that involves renaming of an internalized formula; also, the handling of the projections fst and snd must be done internally.

lemma sats_fst_snd_in_M:

assumes

" $A \in M$ " " $B \in M$ " " $\varphi \in \text{formula}$ " " $p \in M$ " " $l \in M$ "

" $o \in M$ " " $\chi \in M$ " " $\text{arity}(\varphi) \leq 6$ "

shows

" $\{sq \in A \times B .$

$\text{sats}(M, \varphi, [p, l, o, \text{snd}(sq), \text{fst}(sq), \chi])\} \in M$ "

X. CONCLUSIONS AND FUTURE WORK

The ultimate goal of our project is a complete mechanization of forcing allowing for further developments (formalization of the relative consistency of CH), with the long-term hope that working set-theorists will adopt these formal tools as an aid to their research. In the current paper we reported a first major milestone towards that goal; viz. a formal proof in Isabelle/ZF of the satisfaction by generic extensions of most of the ZF axioms.

We cannot overstate the importance of following the sharp and detailed presentation of forcing given by Kunen [17]. In

fact, it helped us to delineate the *thematic* aspects of our formalization; i.e. the handling of all the theoretical concepts and results in the subject and it informed the structure of locales organizing our development. This had an impact, in particular, in the formal statement of the Fundamental Theorems. We consider that the writing of the `forcing_thms` locale, though only taking a few lines of code, is the second most important achievement of this work, since there is no obvious reference from which to translate this directly. The accomplishment of the formalizations of Separation and Powerset are, in a sense, certificates that the locale of the Fundamental Theorems was set correctly.

Replacement is the only one axiom that we have not yet addressed (see below) and Infinity was proved under two extra assumptions on the ground model; when we develop a full-fledged interface between `ctms` of *ZF* and the locales providing recursive constructions from Paulson’s `ZF-Constructible` session, the same current proof will hold with no extra assumptions. The same goes for the results $M \subseteq M[G]$ and $G \in M[G]$.

The Replacement Axiom, however, requires some more work to be done. In Kunen it requires a relativized version (i.e., showing that it holds for M) of the *Reflection Principle*. In order to state this meta-theoretic result by Montague, recall that an equivalent formulation of the Foundation Axiom states that the universe of sets can be decomposed in a transfinite, cumulative hierarchy of sets:

Theorem 8. *Let $V_\alpha := \bigcup\{\mathcal{P}(V_\beta) : \beta < \alpha\}$ for each ordinal α . Then each V_α is a set and $\forall x.\exists\alpha. \text{Ord}(\alpha) \wedge x \in V_\alpha$.*

Theorem 9 (Reflection Principle). *For every finite $\Phi \subseteq ZF$, *ZF* proves: “There exist unboundedly many α such that $V_\alpha \models \Phi$.”*

It is obvious that we can take the conjunction of Φ and state Theorem 9 for a single formula, say φ . The schematic nature of this result hints at a proof by induction on formulas, and hence it must be shown internally. It is to be noted that Paulson [22] also formalized the Reflection principle in *Isabelle/ZF*, but it is not clear if the relativized version follows directly from it. Neeman [20] has an alternative proof for Replacement that might avoid Reflection, for only the relativization of the cumulative hierarchy is needed; it is, nevertheless, a non-trivial task.

This is an appropriate point to insist that the internal/external dichotomy has been a powerful agent in the shaping of our project. This tension was also pondered by Paulson in his formalization of Gödel’s constructible universe [25]; after choosing a shallow embedding of *ZF*, every argument proved by induction on formulas (or functions defined by recursion) should be done using internalized formulas. Working on top of Paulson’s library, we prototyped the formula-transformer `forces`, which is defined for internalized formulas, and this affects indirectly the proof of the Separation Axiom (despite the latter is not by induction). The proof of Replacement also calls for internalized formulas, because one

needs a general version of the Reflection Principle (since the formula φ involved depends on the specific instance of Replacement being proved).

An alternative road to internalization would be to redevelop absoluteness results in a more structured metatheory that already includes a recursively defined type of first order formulas. Needless to say, this change to a deeper-embedding comprises an extensive re-engineering.

A secondary, more prosaic, outcome of this project is to precisely assess which assumptions on the ground model M are needed to develop the forcing machinery. The obvious are transitivity and M being countable (but keep in mind Lemma 1); the first because many absoluteness results follows from this, the latter for the existence of generic filters. A more anecdotal one is that to show that an instance of Separation with at most two parameters holds in $M[G]$, one needs to assume a particular six-parameter instance in M (four extra parameters can be directly blamed on `forces`). The purpose of identifying those assumptions is to assemble in a locale the specific (instances of) axioms that should satisfy the ground model in order to perform forcing constructions; this list will likely include all the instances of Separation and Replacement that are needed to satisfy the requirements of the locales in the `ZF-Constructible` session.

We have already commented on our hacking of `ZF-Constructible` to maximize its modularity and thus the re-usability in other formalizations. We think it would be desirable to organize it somewhat differently: a trivial change is to catalog in one file all the internalized formulas. A more conceptual modification would be to start out with an even more basic locale that only assumes M to be a non-empty transitive class, as many absoluteness results follow from this hypothesis. Furthermore, as Paulson comments in the sources, it would have been better to minimize the use of the Powerset Axiom in locales and proofs. There are useful natural models that satisfy a sub-theory of *ZF* not including Powerset, and to ensure a broader applicability, it would be convenient to have absoluteness results not assuming it. We plan to contribute back to the official distribution of *Isabelle/ZF* with a thorough revision of the development of constructibility.

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APPENDIX
A SHORT OVERVIEW OF OUR DEVELOPMENT

In this appendix we succinctly describe the contents of each file. We include in Figure 1 a dependency graph of our formalization. The theories on a grayish background are directly from Paulson; we highlight with blue/cyan those of Paulson that we modified. We have developed from scratch the rest, in white.

Nat_Miscellanea Miscellaneous results for naturals, mostly needed for renamings.

Renaming Renaming of internalized formulas, see Section IV.

Pointed_DC A pointed version of the Principle of Dependent Choices.

Recursion_Thms Enhancements about recursively defined functions.

Forcing_Notions Definition of Posets with maximal element, filters, dense sets. Proof of the Rasiowa-Sikorski Lemma.

Forcing_Data Definition of the locales: (i) *M_ZF* satisfaction of axioms; and (ii) *forcing_data* extending the previous one with *forcing_notion*, transitivity, and being countable.

Interface Instantiation of locales *M_trivial* and *M_basic* for every instance of *Forcing_Data*.

Names Definitions of *check*, *val*, and the generic extension. Various results about them.

Forcing_Theorems Specification of fundamental theorems of forcing, see Section VIII.

***_Axiom** Proof of the satisfaction of the corresponding axiom in the generic extension.

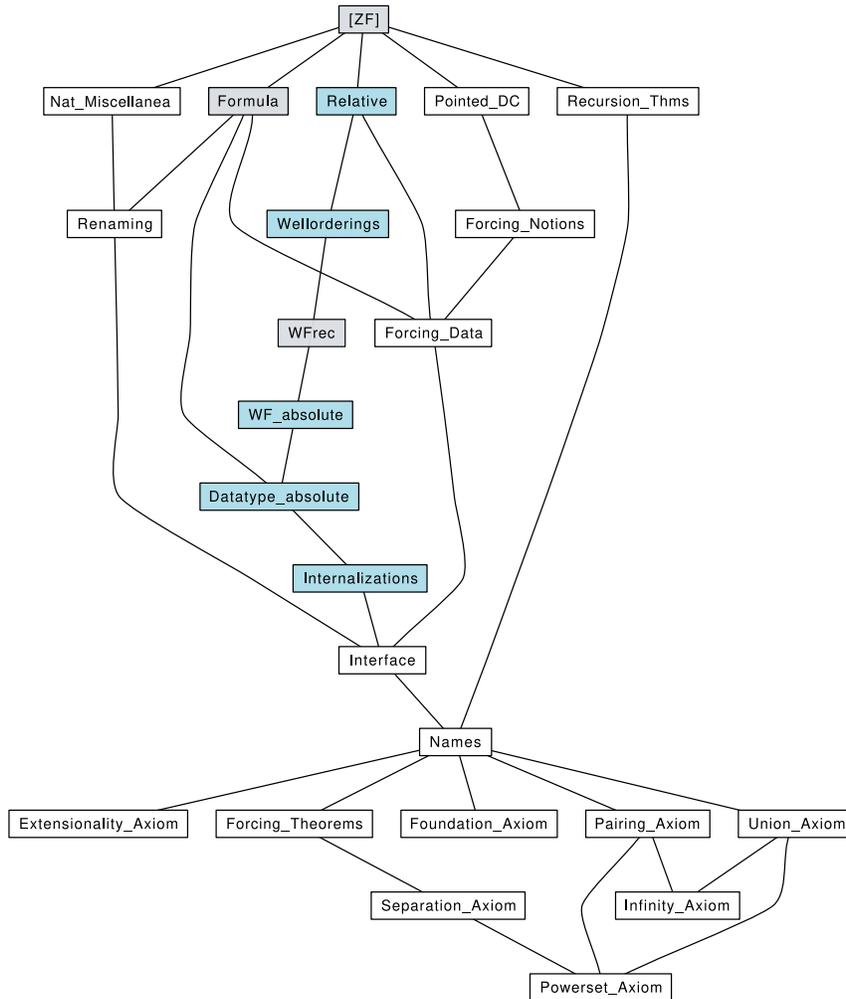


Fig. 1. Dependency graph of the Separation session.