Logics for Markov Decision Processes

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Labelled Transition Systems (LTS)

A toy model

\[ \langle S, L, T \rangle \text{ such that } T_a : S \rightarrow \text{Pow}(S) \text{ for each } a \in L. \]
Labelled Transition Systems (LTS)

\[ \langle S, L, T \rangle \text{ such that } T_a : S \rightarrow \text{Pow}(S) \text{ for each } a \in L. \]

Zig-zag morphism

A surjective \( f : S \rightarrow S' \) such that for all \( a \in L \) and every \( s \in S \),
\[ \text{Pow}(f) \circ T_a = T'_a \circ f. \]
Labelled Transition Systems (LTS)

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\[\text{Pow}(f) \circ T_a = T'_a \circ f.\]

We say that \( s \) simulates \( t \) because \( s \) can perform every “sequence of actions” that \( t \) can.
Simulation and Bisimulation on LTS

**Simulation**

It is a relation $R$ such that if $s_1 \rightarrow t_1$ and $t_1 \xrightarrow{a} t_2$ then there is $s_2$ such that $s_1 \xrightarrow{a} s_2$ and $s_2 \rightarrow t_2$. In that case we say that $s_1$ simulates $s_2$. 
Simulation and Bisimulation on LTS

Simulation

It is a relation $R$ such that if $s_1 R t_1$ and $t_1 \xrightarrow{a} t_2$ then there is $s_2$ such that $s_1 \xrightarrow{a} s_2$ and $s_2 R t_2$. In that case we say that $s_1$ simulates $s_2$.

Bisimulation

It is a symmetric simulation. We’ll say that $s_1$ is bisimilar to $t_1$ if there exists a bisimulation $R$ such that $s_1 R t_1$.

Note: Bisimulation is finer than “double simulation”. That’s to say, if $s_1$ is bisimilar to $t_1$, then $s_1$ simulates $t_1$ and $t_1$ simulates $s_1$, but not conversely.
Coalgebraic presentation of processes and bisimulation

One categorical counterpart of a relation is a span of morphisms

Bisimilarity (span)
Coalgebraic presentation of processes and bisimulation

One categorical counterpart of a relation is a span of morphisms

Bisimilarity (span)

Behavioral equivalence (cospan)

There is a correspondence between cospans and logics
Coalgebraic presentation of processes and bisimulation

One categorical counterpart of a relation is a *span* of morphisms

**Bisimilarity (span)**

\[ \begin{array}{ccc}
S & \xrightarrow{f} & S_1 \\
\downarrow & & \downarrow \\
S & \xrightarrow{g} & S_2 \\
\end{array} \]

**Behavioral equivalence (cospan)**

\[ \begin{array}{ccc}
S_1 & \xrightarrow{\rightarrow} & S_2 \\
\downarrow & & \downarrow \\
T & \xleftarrow{\rightarrow} & T \\
\end{array} \]

There is a correspondence between cospans and logics

**Semipullbacks**

A category *has semipullbacks* if every cospan can be completed to a commutative diagram with a span.
Coalgebraic presentation of processes and bisimulation

One categorical counterpart of a relation is a \textit{span} of morphisms

\begin{center}
\begin{tikzpicture}

\t\node (S) at (0,0) {$S$};
\t\node (S1) at (-1.5,-1) {$S_1$};
\t\node (S2) at (1.5,-1) {$S_2$};

\t\draw[->] (S) to node[above] {$f$} (S1);
\t\draw[->] (S) to node[above] {$g$} (S2);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}

\t\node (S1) at (-1.5,-1) {$S_1$};
\t\node (S2) at (1.5,-1) {$S_2$};
\t\node (T) at (0,0) {$T$};

\t\draw[->] (S1) to node[above] {} (T);
\t\draw[->] (S2) to node[above] {} (T);
\end{tikzpicture}
\end{center}

There is a correspondence between cospans and logics

Semipullbacks

A category \textit{has semipullbacks} if every cospan can be completed to a commutative diagram with a span.

It is the \textbf{Amalgamation Property} in the opposite category.
Logics for Bisimulation

Hennessy-Milner Logic (HML)

\[ \varphi \equiv \top \mid \neg \varphi \mid \bigwedge_{i} \varphi_{i} \mid \langle a \rangle \psi \]
Simulation and Bisimulation on LTS

**Simulation**

It is a relation $R$ such that if $s_1 R t_1$ and $t_1 \xrightarrow{a} t_2$ then there is $s_2$ such that $s_1 \xrightarrow{a} s_2$ and $s_2 R t_2$. In that case we say that $s_1$ simulates $s_2$.

**Bisimulation**

It is a **symmetric** simulation. We’ll say that $s_1$ is **bisimilar** to $t_1$ if there exists a bisimulation $R$ such that $s_1 R t_1$.

“$t_1$ can make an $a$-transition after which a $c$-transition is not possible”.

$t_1 \models \langle a \rangle \neg \langle c \rangle \top$

$s_1 \not\models \langle a \rangle \neg \langle c \rangle \top$
Logics for Bisimulation

Hennessy-Milner Logic (HML)

$$\varphi \equiv \top | \neg \varphi | \bigwedge_{i} \varphi_i | \langle a \rangle \psi$$

Logical Characterization of Bisimulation

Two states in a LTS are bisimilar iff they satisfy the same HML formulas.
Labelled Markov Processes (LMP) and Non Determinism

LMP (Desharnais et al.)

\( \langle S, S, L, t \rangle \) such that \( t_a(s) \in \mathbf{P}(S) \) for each \( s \in S \) and \( a \in L \), where

- \( \langle S, S \rangle \) is a measurable space;
- \( \mathbf{P}(S) \) is the space of (sub)probability measures over \( \langle S, S \rangle \);
- \( t_a : S \rightarrow \mathbf{P}(S) \) is measurable.

NLMP (D'Argenio and Wolovick)

\( \langle S, S, L, T \rangle \) such that \( T_a(s) \subseteq \mathbf{P}(S) \) for each \( s \in S \) and \( a \in L \), where:

- \( \langle S, S \rangle \), \( \mathbf{P}(S) \) as before;
- For each \( s \), \( T_a(s) \) is measurable. I.e., \( T_a : S \rightarrow \mathbf{P}(S) \).
Labelled Markov Processes (LMP) and Non Determinism

LMP (Desharnais et al.)
\[
\langle S, S, L, t \rangle \text{ such that } t_a(s) \in \mathbf{P}(S) \text{ for each } s \in S \text{ and } a \in L, \text{ where}
\]
- \langle S, S \rangle \text{ is a measurable space;}
- \mathbf{P}(S) \text{ is the space of (sub)probability measures over } \langle S, S \rangle;
- t_a : S \rightarrow \mathbf{P}(S) \text{ is measurable.}

NLMP (D’Argenio and Wolovick)
\[
\langle S, S, L, T \rangle \text{ such that } T_a(s) \subseteq \mathbf{P}(S) \text{ for each } s \in S \text{ and } a \in L, \text{ where:}
\]
- \langle S, S \rangle, \mathbf{P}(S) \text{ as before;}
- For each } s, \text{ } T_a(s) \text{ is measurable. I.e., } T_a : S \rightarrow \mathbf{P}(S).
- T_a : S \rightarrow \mathbf{P}(S) \text{ is a measurable map.
A pinch of Descriptive Set Theory: Analytic Spaces

Definition

An analytic topological space is the continuous image of a Borel set (v.g., of reals).
Analytic Spaces and Unique Structure

A pinch of Descriptive Set Theory: Analytic Spaces

**Definition**

An *analytic* topological space is the continuous image of a Borel set (v.g., of reals).

An measurable space is *analytic* if it is isomorphic to $\langle A, B(A) \rangle$ for some analytic topological space $A$. 
A pinch of Descriptive Set Theory: Analytic Spaces

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**Examples**

- The convex hull of a Borel set in $\mathbb{R}^n$;
- The relation of isomorphism between countable structures.
Analytic Spaces and Unique Structure

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An *analytic* topological space is the continuous image of a Borel set (v.g., of reals).

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**Examples**

- The convex hull of a Borel set in $\mathbb{R}^n$;
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**Unique Structure Theorem**

If a sub-$\sigma$-algebra $\mathcal{S} \subseteq \mathcal{B}(A)$ is countably generated and separates points, then it is $\mathcal{B}(A)$. 
Introduction

Labelled Markov Processes (LMP) and its Non Deterministic version

Results

Future Work

Proving Completeness

Logics for bisimulation on LMP

HML_q (Larsen and Skou, Danos et al.)

ϕ ≡ T | ϕ_1 ∧ ϕ_2 | ⟨a⟩_q ϕ, q ∈ Q
Proving Completeness

Logics for bisimulation on LMP

HML\_q (Larsen and Skou, Danos et al.)

\[ \varphi \equiv \top \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle_q \varphi, \quad q \in Q \]

Logical Characterization of Bisimulation for LMP (Danos et al.)

Two states in a LMP \( \langle S, S, L, t \rangle \) with \( \langle S, S \rangle \) analytic are bisimilar iff they satisfy the same HML\_q formulas.
Proving Completeness

Logics for bisimulation on LMP

\[ HML_q (\text{Larsen and Skou, Danos et al.}) \]
\[ \varphi \equiv \top \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle_q \varphi, \quad q \in \mathbb{Q} \]

Logical Characterization of Bisimulation for LMP (Danos et al.)

Two states in a LMP \( \langle S, S, L, t \rangle \) with \( \langle S, S \rangle \) analytic are bisimilar iff they satisfy the same \( HML_q \) formulas.

Proof Strategy (D’Argenio, Celayes, PST)

This results holds for every process with an analytic state space and a logic \( L \) that satisfies: 1) \( L \) it contains \( \top \) and \( \land \); 2) for every \( \varphi \in L \), \( \llbracket \varphi \rrbracket \) is measurable; 3) \( L \) is countable; and 4) \( L \) separates transitions “locally”.

\[ \top \land \varphi \]
Logics for non-deterministic processes

Logics for bisimulation on LMP

$L_f$ (D’Argenio et al.)

\[ \varphi \equiv \top \lor \varphi_1 \land \varphi_2 \lor \langle a \rangle \{ \varphi_i, p_i \}_{i=1}^n, \quad p_i \in \mathbb{Q}, \ n \in \mathbb{N} \]
Logics for non-deterministic processes

Logics for bisimulation on LMP

$L_f$ (D’Argenio et. al)

\[
\phi \equiv \top \mid \phi_1 \land \phi_2 \mid \langle a \rangle \{\phi_i, p_i\}_{i=1}^n, \quad p_i \in Q, \ n \in \mathbb{N}
\]

The proof strategy immediately gives

Logical Characterization of Bisimulation for image finite NLMP

Two states in a image finite NLMP $\langle S, S, L, t \rangle$ with $\langle S, S \rangle$ analytic are bisimilar iff they satisfy the same $L_f$ formulas.
Logics for non-deterministic processes

Logics for bisimulation on LMP

$\mathcal{L}_f$ (D’Argenio et. al)

$$\varphi \equiv \top \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle \{ \varphi_i, p_i \}_{i=1}^n, \quad p_i \in \mathbb{Q}, \; n \in \mathbb{N}$$

The proof strategy immediately gives

Logical Characterization of Bisimulation for image finite NLMP

Two states in a image finite NLMP $\langle S, S, L, t \rangle$ with $\langle S, S \rangle$ analytic are bisimilar iff they satisfy the same $\mathcal{L}_f$ formulas.

$\Delta$ (D’Argenio et. al)

$$\varphi \equiv \top \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle \psi$$

$$\psi \equiv \bigvee_{i \in I} \psi_i \mid \neg \psi \mid [\varphi]_{\geq q}$$
Some counterexamples

Analiticity is necessary

The category of LMP over arbitrary measurable spaces does not have semipullbacks and HMLₜ does not characterize bisimilarity (PST, *Inf & Comp.* 209 2011)

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Analiticy is necessary

The category of LMP over arbitrary measurable spaces does not have semipullbacks and $\text{HML}_q$ does not characterize bisimilarity (PST, *Inf& Comp.* 209 2011).

At least image-countable is necessary

Logics for bisimulation on LMP

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\[ \varphi \equiv \top \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle_q \varphi, \quad q \in \mathbb{Q} \]

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Two states in a LMP \( \langle S, S, L, t \rangle \) with \( \langle S, S \rangle \) analytic are bisimilar iff they satisfy the same HML_q formulas.

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This results holds for every process with an analytic state space and a logic \( \mathcal{L} \) that satisfies: 1) \( \mathcal{L} \) it contains \( \top \) and \( \land \); 2) for every \( \varphi \in \mathcal{L}, \llbracket \varphi \rrbracket \) is measurable; 3) \( \mathcal{L} \) is countable; and 4) \( \mathcal{L} \) separates transitions “locally”.
Future Work

- To decide whether there is a nice logical characterization of bisimulation for countable NLMP. **Is there a countable logic for countable LTS?**
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- To decide whether there is a nice logical characterization of bisimulation for countable NLMP. Is there a countable logic for countable LTS?
- If possible, to extend the logical characterization to Radon spaces \( \langle S, \mathcal{S} \rangle \) (i.e., \( \mathcal{S} \subseteq \text{universally measurable sets} \)).
Thank You!
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