Logics with Copy and Remove

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Abstract. We propose a logic with the dynamic modal operators copy and remove. The copy operator replicates a given model, and the remove operator removes paths in a given model. We show that the product update by an action model (with Boolean pre-conditions) in dynamic epistemic logic decomposes in copy and remove operations. We also show that copy and remove operators (of path of length 1) can be expressed by action models. We investigate the complexity of the satisfiability problem of syntactic fragments of the logic with copy and remove operations.

Keywords: modal logic, dynamic epistemic logic, complexity, expressivity.

1 Introduction

In modal logic we interpret a modal operator by way of an accessibility relation in a given model. Over the past decades some logics have been proposed in which the modality is, instead, interpreted by a transformation of the model. In such logics the modality can be seen as interpreted by a binary relation between pointed Kripke models, where the second argument of the relation is the transformed model. We could mention sabotage logic here [10], wherein states or arrows are deleted from a model. Or we could mention dynamic epistemic logics [13] that focus on such model changing operators in view of modeling change of knowledge or belief (the standard interpretation for the basic modalities in that setting). In [1,2,6] a new line of contributions to model-transforming logics, motivated by van Benthem's sabotage logic is developed. Our contribution advances that last line of work, while linking it to dynamic epistemic logics.

Action model logic ($\mathcal{AML}$) [4] is a well-known dynamic epistemic logic to model information change. Action model logic is an extension of basic epistemic logic with a dynamic modal operator for the execution of actions. This operator is parameterized by an action model, a semantic object which typically models a multi-agent information changing scenario. These actions models are treated as syntactic objects in modal operators. Action models are complex structures,

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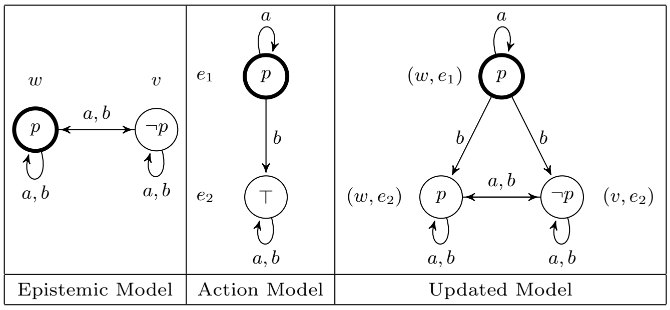

Fig. 1. Agent a privately learns that p.

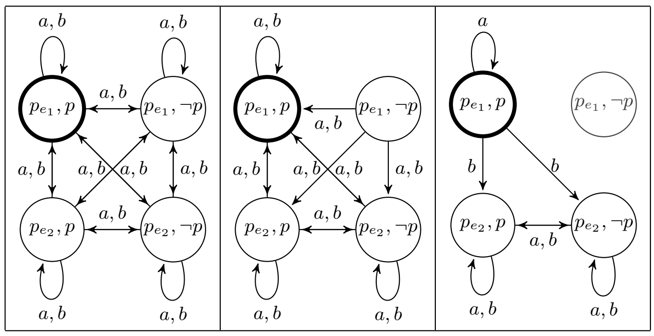

Fig. 2. The result of one copy and two remove operations on the epistemic model of Figure 1, again resulting in the same (bisimilar) updated model.

which also leads to high computational complexity (deciding model checking is PSPACE-complete, while deciding satisfiability is NEXPTIME-complete [3]).

In this contribution we propose modal logics with primitive actions called copy and remove. We investigate some of their model theoretic properties and their complexity, and, as an example of what one can do with such logics, we give an embedding of action model logic into our logic: we show that every action model (with propositional pre-conditions) can be simulated by a combination of the copy and remove operators. This is in line with the previously known result that, on the class of finite models, action model execution corresponds to model restriction ('remove') on a bisimilar copy ('copy') of the initial model [11]. The delete we propose is akin to the generalized arrow updates of [9], continuing the work started in [8], that are also known to have equal expressivity as action model logic. But the copy and remove operators we propose are more procedural, whereas these mentioned results are more of a declarative nature.

In Figure 1 we show an epistemic model (a Kripke model), an action model, and the result of executing that action model in that epistemic model. The epistemic model represents that agents a and b are uncertain whether an atomic proposition p is true (and that they have common knowledge of that uncertainty). The actual world, or designated state, of the model is where p is true (shown with a thick circle in the Figure). The action model represents that agent a learns that p is true, whereas agent b (incorrectly) believes that nothing happens—of which a is aware. In short: a privately learns that p. In action models, the valuations of propositional variables are replaced by pre-conditions, in this case p and $\top$ (the formula that is always true). Action models update Kripke models by mean of a restricted modal product, where the domain is limited to the state-action pairs where the pre-conditions of the actions hold. Therefore, there are only three (and not four) such pairs in the updated model: the pair (w, e_1) is missing as the pre-condition of e_1, the formula p, is not true in the state w. The arrows in the product are updated according to the principle that there is a (labeled) arrow between two state-action pairs if there was such an arrow linking both the first arguments and the second arguments. One can now establish that in the resulting model a knows that p (there is only an a-arrow from w to itself), whereas b still believes that a, b are ignorant whether p.

By means of the copy and remove actions of the logics that we propose, we can alternatively describe the effect of this action model. This is depicted in Figure 2. First, we replicate the original epistemic model as many times as there are actions in the action model (twice in this case). We identify each copy with a (fresh) propositional variable corresponding to an action in the action model (e.g., p_{e_1} corresponds to e_1). Thus we obtain the leftmost model in Figure 2. Then, we first remove all the edges (arrows) that point to state-action alternatives wherein the action cannot be executed in the state. Finally, between the remaining state-action pairs we remove all edges that are ruled out according to the accessibility relation in the action model. Thus we obtain the rightmost model in Figure 2.

2 Copy and Remove

In this section we introduce $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$, a language which can remove edges and create copies of a model.

Definition 1 (Syntax). *Given* PROP*, an infinite and countable set of propositional symbols, and* AGT*, a finite set of agents, let us define the set* FORM *of* $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$*-formulas, together with a set* PATH *of path expressions.*

$$\mathsf{FORM} ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \Diamond_a \varphi \mid \mathsf{rm}(\pi)\varphi \mid \mathsf{cp}(\bar{p}, q)\varphi,$$

where $\bar{p} = \langle p_1, \ldots, p_n \rangle$ *is any finite sequence of propositional symbols (all distinct among them) that do not appear in any occurrence of* cp *in* φ*,* $q \in \bar{p}$*,* $a \in \mathsf{AGT}, \varphi, \varphi' \in \mathsf{FORM}$*, and* $\pi \in \mathsf{PATH}$*.*

$$\mathsf{PATH} ::= a \mid \pi; \pi' \mid \varphi?,$$

where $a \in \mathsf{AGT}$*,* $\pi, \pi' \in \mathsf{PATH}$ *and* φ *is a Boolean formula.*

We also define the following syntactic fragments: $\mathcal{ML}(\mathsf{cp})$, the fragment with the cp operator but without rm; $\mathcal{ML}(\mathsf{rm})$, the fragment with the rm operator but without cp; $\mathcal{ML}(\mathsf{rm}^-)$, the fragment with rm with path expressions only of the form $\pi = \varphi?; a; \psi?$ but without cp; and $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$, the fragment with rm with path expressions only of the form $\pi = \varphi?; a; \psi?$ and with cp.

Definition 2 (Models). *A model $\mathcal{M}$ is a triple $\mathcal{M} = \langle W, R, V\rangle$, where W is a non-empty set; $R \subseteq \mathsf{AGT} \times W^2$ is an accessibility relation (we will often write R_a to refer to the set $\{(w, v) \in W^2 \mid (a, w, v) \in R\}$); and $V : \mathsf{PROP} \to \mathcal{P}(W)$ is a valuation. A pair $\mathcal{M}, w$ where w is a state in $\mathcal{M}$ is called a pointed model.*

We represent a path as a sequence $w_0 a_0 w_1 a_1 \dots w_{n-1} a_{n-1} w_n$ where w_i are states and a_i are agents. Let us now define the set $\mathcal{P}^{\mathcal{M}}(\pi)$ of π-paths in the model $\mathcal{M}$ by induction on π. $\mathcal{P}^{\mathcal{M}}(a)$ contains paths representing a-edges. $\mathcal{P}^{\mathcal{M}}(\pi; \pi')$ contains concatenations of a π-path and a π'-path. In such a concatenation, the last state w of the π-path has to be the first state of the π'-path. $\mathcal{P}^{\mathcal{M}}(\varphi?)$ contains paths of length 0, made of one state, which satisfies φ.

Definition 3 (Paths and Updated Models). *Let $\mathcal{M} = \langle W, R, V\rangle$ a model and $\pi \in \mathsf{PATH}$. We define the set of π-paths $\mathcal{P}^{\mathcal{M}}(\pi)$ of $\mathcal{M}$ inductively as*

$$\begin{aligned}
\mathcal{P}^{\mathcal{M}}(a) &= \{wau \mid (w, u) \in R_a\}\\
\mathcal{P}^{\mathcal{M}}(\pi; \pi') &= \{SwS' \mid Sw \in \mathcal{P}^{\mathcal{M}}(\pi) \text{ and } wS' \in \mathcal{P}^{\mathcal{M}}(\pi')\}\\
\mathcal{P}^{\mathcal{M}}(\varphi?) &= \{w \mid \mathcal{M}, w \models \varphi\}.
\end{aligned}$$

Let $a \in \mathsf{AGT}$, we define $edges_a(P)$ that is the set of a-edges of the path P. Formally, $edges_a(P) = \{(a, w, u) \mid wau \text{ is a subsequence of } P\}$.

Given a model $\mathcal{M} = \langle W, R, V\rangle$, a path expression π, and $\bar{p} = \langle p_1, \dots, p_n\rangle$, we define the updated models

$$\begin{aligned}
&\mathcal{M}_{\mathsf{rm}(\pi)} = \langle W, R_{\mathsf{rm}(\pi)}, V\rangle, \text{ where}\\
&\quad R_{\mathsf{rm}(\pi)} = R \setminus \textstyle\bigcup_{a \in \mathsf{AGT}, P \in \mathcal{P}^{\mathcal{M}}(\pi)} edges_a(P)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}_{\mathsf{cp}(\bar{p})} = \langle W_{\mathsf{cp}(\bar{p})}, R_{\mathsf{cp}(\bar{p})}, V_{\mathsf{cp}(\bar{p})}\rangle, \text{ where}\\
&\quad W_{\mathsf{cp}(\bar{p})} = \{(w, q) \mid w \in W \text{ and } q \in \bar{p}\}\\
&\quad R_{\mathsf{cp}(\bar{p})} = \{(a, (w, q), (w', q')) \mid (a, w, w') \in R\}\\
&\quad V_{\mathsf{cp}(\bar{p})}(p) = \{(w, q) \mid w \in V(p)\} \quad \text{for } \; p \neq q\\
&\quad V_{\mathsf{cp}(\bar{p})}(q) = \{(w, q) \mid w \in W\}.
\end{aligned}$$

Now we can define the semantics of the operators introduced in Definition 1.

Definition 4 (Semantics). *Given a pointed model $\mathcal{M}, w$ and a formula φ we say that $\mathcal{M}, w$ satisfies φ, and write $\mathcal{M}, w \models \varphi$, when*

$$\begin{array}{ll}
\mathcal{M}, w \models p & \text{iff } \; w \in V(p)\\
\mathcal{M}, w \models \neg\varphi & \text{iff } \; \mathcal{M}, w \not\models \varphi\\
\mathcal{M}, w \models \varphi \land \psi & \text{iff } \; \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi\\
\mathcal{M}, w \models \Diamond_a \varphi & \text{iff } \; \text{for some } v \in W \text{ s.t. } (w, v) \in R_a, \; \mathcal{M}, v \models \varphi\\
\mathcal{M}, w \models \mathsf{rm}(\pi)\varphi & \text{iff } \; \mathcal{M}_{\mathsf{rm}(\pi)}, w \models \varphi\\
\mathcal{M}, w \models \mathsf{cp}(\bar{p}, q)\varphi & \text{iff } \; \mathcal{M}_{\mathsf{cp}(\bar{p})}, (w, q) \models \varphi.
\end{array}$$

φ is satisfiable if for some pointed model $\mathcal{M}, w$ we have $\mathcal{M}, w \models \varphi$. When the left side is empty, $\models \varphi$ means that φ holds in any model. We further define $\mathsf{cp}(\bar{p})\varphi$ as an abbreviation for $\bigwedge_{q \in \bar{p}} \mathsf{cp}(\bar{p}, q)\varphi$.

Bisimulation is a classical notion introduced to investigate the expressive power of modal languages. The conditions required for $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$ turn out to be very natural: paths deleted via rm traversing a particular state are characterized by the information in successors and predecessors of such point. Hence, it is enough to consider the conditions for the basic temporal logic $\mathcal{ML}(\Diamond^{-1})$ (see [5]):

(Atomic Harmony) for all $p \in \mathsf{PROP}$, $w \in V(p)$ iff $w' \in V'(p)$;
(Zig) if $(w, v) \in R_a$ then for some v', $(w', v') \in R'_a$ and vZv';
(Zag) if $(w', v') \in R'_a$ then for some v, $(w, v) \in R_a$ and vZv'.
(Zig^{-1}) if $(v, w) \in R_a$ then for some v', $(v', w') \in R'_a$ and vZv';
(Zag^{-1}) if $(v', w') \in R'_a$ then for some v, $(v, w) \in R_a$ and vZv'.

Let $\leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})}$ refer to bisimulations for the language $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$. We can prove that $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$-bisimilar models satisfy the same formulas.

Theorem 1 (Invariance under bisimulation.). *For all $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$-formula φ, we have $\mathcal{M}, w \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$ implies $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.*

From the tree model property for $\mathcal{ML}(\Diamond^{-1})$ it immediately follows:

Corollary 1. *The language $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$ has the tree model property.*

3 Relation between action models and $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$

In this section we start by formally introducing action model logic and then define an embedding into $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$. We restrict ourselves to the case where preconditions in action models are Boolean.

Definition 5 (Action Models). *Let $\mathcal{B}$ be the set of Boolean formulas over certain set PROP of propositional symbols. An action model $\mathcal{E}$ is a structure $\mathcal{E} = \langle E, \rightarrow, \mathsf{pre}, \mathsf{post}\rangle$, where E is a non-empty finite set whose elements are called action points; for each $a \in \mathsf{AGT}$, $\rightarrow (a) \subseteq E \times E$ is an equivalence relation (we will often write $\rightarrow_a$ rather than $\rightarrow (a)$); $\mathsf{pre} : E \rightarrow \mathcal{B}$ is a pre-condition function; and $\mathsf{post} : E \rightarrow \mathsf{PROP} \rightarrow \{\top, \bot\}$ is a post-condition function. Let e be an action point in $\mathcal{E}$, the pair $(\mathcal{E}, e)$ is a pointed action model.*

Action models in action model logic appear as modalities. We will call $\mathcal{AML}$ the fragment where action models have only pre-conditions, i.e., action models of the shape $\langle E, \rightarrow, \mathsf{pre}\rangle$, and use $\mathcal{AML}^+$ for the full language.

Definition 6 (Syntax). *Let PROP be a countable, infinite set of propositional symbols and AGT a finite set of agent symbols. The set FORM of formulas of $\mathcal{AML}$ and $\mathcal{AML}^+$ over PROP and AGT is defined as:*

$$\mathsf{FORM} ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond_a \varphi \mid [\alpha]\varphi,$$

where $p \in \mathsf{PROP}$*,* $a \in \mathsf{AGT}$*,* $\varphi, \psi \in \mathsf{FORM}$ *and* $\alpha \in \mathsf{ACT}$*. The set of actions* ACT *is defined as* $\mathsf{ACT} ::= \mathcal{E}, e \mid \alpha \cup \beta$*, with* $\mathcal{E}, e$ *an action pointed model and* $\alpha, \beta \in \mathsf{ACT}$*.* $\langle \alpha \rangle \varphi$ *is a shorthand for* $\neg[\alpha]\neg\varphi$*.*

Definition 7 (Semantics). *Given an epistemic pointed model* $\mathcal{M}, w$ *with* $\mathcal{M} = \langle W, R, V \rangle$*, an action pointed model* $\mathcal{E}, e$ *with* $\mathcal{E} = \langle E, \rightarrow, \mathsf{pre}, \mathsf{post} \rangle$*, and a formula* φ *we say that* $\mathcal{M}, w \models \varphi$ *when*

$$\begin{array}{ll} \mathcal{M}, w \models [\alpha]\varphi & \textit{iff for all } \mathcal{M}', w' \textit{ s.t. } \mathcal{M}, w[\![\alpha]\!]\mathcal{M}', w' \textit{ we have } \mathcal{M}', w' \models \varphi \\ \mathcal{M}, w[\![\mathcal{E}, e]\!]\mathcal{M}', w' & \textit{iff } \mathcal{M}, w \models \mathsf{pre}(e) \textit{ and } \mathcal{M}', w' = (\mathcal{M} \otimes \mathcal{E}), (w, e) \\ [\![\alpha \cup \beta]\!] & = [\![\alpha]\!] \cup [\![\beta]\!]. \end{array}$$

where $(\mathcal{M} \otimes \mathcal{E})$ *is defined as* $\langle W', R', V' \rangle$*, with:*

$$\begin{array}{ll} W' & = \{(v, d) \in W \times E \mid \mathcal{M}, v \models \mathsf{pre}(d)\} \\ ((v, d), (u, f)) \in R'_a & \textit{iff } (v, u) \in R_a \textit{ and } d \rightarrow_a f \\ V'(p) & = \{(v, d) \mid \mathcal{M}, v \models \mathsf{post}(e)(p)\}. \end{array}$$

If $\mathcal{E}$ *does not have post-conditions then* $V'(p) = \{(v, d) \mid v \in V(p)\}$*.*

We now show how to embed $\mathcal{AML}$ into $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$. First, define the shorthand $\mathsf{rm}(\pi_1 \odot \pi_2)\varphi$ for $\mathsf{rm}(\pi_1)\mathsf{rm}(\pi_2)\varphi$. Notice that if π_1 and π_2 are paths of size 1, and given that we are only considering Boolean tests, then $\odot$ is commutative.

Definition 8. *Let* $\mathcal{E} = \langle E, \rightarrow, \mathsf{pre} \rangle$ *be an action model with* $E = \{e_1, \ldots, e_n\}$*. We define the translation* Tr *from* $\mathcal{AML}$*-formulas to* $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$*-formulas as:*

$$\mathsf{Tr}([\mathcal{E}, e_1]\varphi) = \mathsf{pre}(e_1) \rightarrow \mathsf{cp}(\langle p_{e_1} \ldots p_{e_n} \rangle)\mathsf{rm}(\rho)\mathsf{rm}(\sigma)\mathsf{Tr}(\varphi),$$

$$\textit{where} \quad \begin{array}{l} \rho \equiv \bigodot_{e_i \in E, a \in \mathsf{AGT}} \top?; a; (p_{e_i} \wedge \neg\mathsf{pre}(e_i))? \\ \sigma \equiv \bigodot_{e_i, e_j \in E, a \in \mathsf{AGT}} p_{e_i}?; a; p_{e_j}? \qquad \textit{if } e_i \not\rightarrow_a e_j. \end{array}$$

Tr *commutes with all other formulas.*

Proposition 1. *Let* φ *be an* $\mathcal{AML}$*-formula, then* φ *and* $\mathsf{Tr}(\varphi)$ *are equivalent.*

Proof (Sketch). The antecedent $\mathsf{pre}(e_1)$ is exactly the same clause as for model updates (considering the pointed action model $\mathcal{E}, e_1$ as the desired update). For each action $e_i \in E$, we consider a propositional symbol p_{e_i}. The operation $\mathsf{cp}(\langle p_{e_1} \ldots p_{e_n} \rangle)$ replicates the original model as many times as actions in E (notice that we can always use isomorphic action models to ensure that the propositional symbols used by cp are new). This operation generates the cartesian product $W \times E$. However, the model $\mathcal{M} \otimes \mathcal{E}$ does not consider the whole cartesian product. To cut the unwanted part of the model we introduce $\mathsf{rm}(\rho)$. The path expression ρ characterizes all the edges we introduced by the previous $\mathsf{cp}(\langle p_{e_1} \ldots p_{e_n} \rangle)$ pointing to p_{e_i}-states which do not satisfy the corresponding $\mathsf{pre}(e_i)$. In the same way that it is done in $\mathcal{AML}$ product updates, we remove all arrows pointing to those states. Once we have constructed the domain, it remains to restrict the obtained accessibility relation. This is done by $\mathsf{rm}(\sigma)$. Remember that $((v,d),(u,f)) \in R'_a$ in $\mathcal{M} \otimes \mathcal{E}$ if and only if $(v,u) \in R_a$ and $d \rightarrow_a f$. The first part trivially holds in the translation, because cp does not introduce edges between copies of elements that were not related in the original model. Then $\mathsf{rm}(\sigma)$ deletes all the a-edges (w_i, w_j) such that in the action model there is no a-edge from e_i to e_j, for all $a \in \mathsf{AGT}$.

The obtained model is not $\mathcal{M}\otimes\mathcal{E}$, but it is bisimilar according to bisimulation for $\mathcal{ML}$, which is the notion used in $\mathcal{AML}$. As a result, they represent the same information for the agents. □

In Figure 2 we see the encoding above applied to a concrete update. The first step of the translation replicates as many copies of the original epistemic model, as actions belonging to the domain of the action model, obtaining the leftmost model. This is done via a copy operation. Next, evaluating $\mathsf{rm}(\rho)$ (defined as in Definition 8), we remove all the edges pointing to states where at the same time p_{e_1} holds and $\mathsf{pre}(e_1)$ does not hold, and we get the model in the center of Figure 2. Last, we need to evaluate $\mathsf{rm}(\sigma)$. This removes those edges that have been added by the copy operation, but are not connected in the original action model. Thereby, we remove all the undesirable accessibility edges, obtaining the rightmost model, which is bisimilar to the updated model of Figure 1 (the state labeled by $\{p_{e_1}, \neg p\}$ is not longer accessible).

We show now that copy and remove can be seen as action models in $\mathcal{AML}^+$. This is valuable, as it demonstrates that action models have a certain decomposition: an action model can be described as the composition of simpler action models. This decomposition can be obtained by translating first into $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ and then considering copy and remove again as basic action models.

Consider the copy action $\mathsf{cp}(\bar{p})$, and let Q be the set of all propositional symbols occurring in $\bar{p}$. The copy operator can be modeled as an action model $\mathcal{E}(\mathsf{cp}(\bar{p})) = \langle E, \rightarrow, \mathsf{pre}, \mathsf{post}\rangle$ such that (for all $q \in E = Q$, $a \in \mathsf{AGT}$):

$$\begin{array}{llll} E &= Q & \mathsf{pre}(q) &= \top \\ \rightarrow_a &= E \times E & \mathsf{post}(q)(q) &= \top \\ & & \mathsf{post}(q)(p) &= \bot \qquad \text{for } p \in Q \setminus \{q\}. \end{array}$$

We note that for all $r \in \mathsf{PROP} \setminus Q$ the value is not affected at the execution of this action, as the finite subset of propositional symbols that is assigned a post-condition is the set Q. Consider the translation $'$: $\mathcal{ML}(\mathsf{cp}) \rightarrow \mathcal{AML}$ such that $(\mathsf{cp}(\bar{p})\phi)' = [\mathcal{E}(\mathsf{cp}(\bar{p}))]\phi'$ and commutes with all other operators. Then:

Proposition 2. *For all $\varphi \in \mathcal{ML}(\mathsf{cp})$, φ and φ' are equivalent.*

Next, we study the remove action. The action model $\mathcal{E}(\mathsf{rm}(\phi?; a; \psi?)) = \langle E, \rightarrow, \mathsf{pre}\rangle$ is defined as

$$\begin{array}{lll} E &= \{00, 10, 01, 11\} & \mathsf{pre}(00) = \neg\phi \wedge \neg\psi, \\ \rightarrow_a &= (E \times E) \setminus \{(10,01),(10,11),(11,01),(11,11)\} & \mathsf{pre}(10) = \phi \wedge \neg\psi, \\ \rightarrow_b &= (E \times E) \quad \text{for all } b \neq a & \mathsf{pre}(01) = \neg\phi \wedge \psi, \\ & & \mathsf{pre}(11) = \phi \wedge \psi. \end{array}$$

This action model corresponds to the operation of removing all $\phi \xrightarrow{a} \psi$ arrows. Consider the translation $'' : \mathcal{ML}(\mathsf{rm}^-) \to \mathcal{AML}$ such that $(\mathsf{rm}(\phi?;a;\psi?)\theta)'' = [\mathcal{E}(\mathsf{rm}(\phi?;a;\psi?))]\theta''$ and commutes with all other operators.

Proposition 3. *For all $\varphi \in \mathcal{ML}(\mathsf{rm}^-)$, φ and φ'' are equivalent.*

4 Complexity of deciding satisfiability

The following result has been proved already in [6]:

Theorem 2. *Deciding if a formula in $\mathcal{ML}(\mathsf{cp})$ is satisfiable is* PSPACE*-complete.*

We will show that $\mathcal{ML}(\mathsf{rm})$ can be translated into $\mathcal{ML}(\Diamond^{-1})$, the basic modal logic $\mathcal{ML}$ with the past operator $\Diamond^{-1}$. It is easy to see that as tests are Boolean, if two tests are consecutive in a path expression (e.g., $\varphi_1?;\varphi_2?$), we can replace them by a single test (e.g., $(\varphi_1 \wedge \varphi_2)?$). If two agents are consecutive in a path expression (e.g., $a_1;a_2$) we can add a trivial test between them (e.g., $a_1;\top?;a_2$). Thus, without loss of generality we assume that all delete operators have the form

$$\mathsf{rm}(\varphi_1?;a_1;\varphi_2?;a_2;\ldots;a_{n-1};\varphi_n?)\psi,$$

where $\varphi_i?$ are arbitrary Boolean formulas, and $a_i \in \mathsf{AGT}$. We introduce reduction axioms to get an $\mathcal{ML}(\Diamond^{-1})$-formula, and conclude that any $\mathcal{ML}(\mathsf{rm})$-formula, is equivalent to an $\mathcal{ML}(\Diamond^{-1})$-formula.

First, let us define the abbreviations $\Diamond_{i,j}$, $\Diamond^{-1}_{i,j}$, for a fix path expression $\pi = \varphi_1?;a_1;\ldots;a_{n-1};\varphi_n?$:

$$\Diamond_{i,j} = \begin{cases} \top & j < i \\ \Diamond_{a_i}\varphi_{i+1} & i = j \\ \Diamond_{a_i}(\varphi_{i+1} \wedge \Diamond_{i+1,j}) & i < j \end{cases} \quad \Bigg| \quad \Diamond^{-1}_{i,j} = \begin{cases} \top & j < i \\ \Diamond^{-1}_{a_i}\varphi_i & i = j \\ \Diamond^{-1}_{a_j}(\Diamond^{-1}_{i,j-1} \wedge \varphi_j) & i < j \end{cases}$$

Now define $rm^\pi_i = \Diamond^{-1}_{1,i-1} \wedge \varphi_i \wedge \Diamond_{i,n-1}$. Informally rm^π_i means "the current state is at position i in a path that matches $\pi = \varphi_1?;a_1;\varphi_2?;a_2;\ldots;a_{n-1};\varphi_n?$ which is going to be deleted". For instance, rm^π_i, $1 \le i \le n$ are defined as:

$$\begin{array}{lll}
rm^\pi_1 & = & \varphi_1 \wedge (\Diamond_{a_1}\varphi_2 \wedge (\Diamond_{a_2}\varphi_3 \ldots \wedge \Diamond_{a_{n-2}}(\varphi_{n-1} \wedge \Diamond_{a_{n-1}}\varphi_n)\ldots)) \\
rm^\pi_2 & = & \Diamond^{-1}_{a_1}\varphi_1 \wedge \varphi_2 \wedge (\Diamond_{a_2}\varphi_3 \ldots \wedge \Diamond_{a_{n-2}}(\varphi_{n-1} \wedge \Diamond_{a_{n-1}}\varphi_n)\ldots) \\
 & \ldots & \\
rm^\pi_{n-1} & = & \Diamond^{-1}_{a_{n-2}}(\Diamond^{-1}_{a_{n-3}}(\ldots(\Diamond^{-1}_{a_1}\varphi_1 \wedge \varphi_2) \wedge \varphi_3)\ldots) \wedge \varphi_{n-1} \wedge \Diamond_{a_{n-1}}\varphi_n \\
rm^\pi_n & = & \Diamond^{-1}_{a_{n-1}}(\Diamond^{-1}_{a_{n-2}}(\ldots(\Diamond^{-1}_{a_1}\varphi_1 \wedge \varphi_2) \wedge \varphi_3 \ldots) \wedge \varphi_{n-1}) \wedge \varphi_n.
\end{array}$$

Lemma 3. *Let $\mathcal{M}{=}\langle W,R,V\rangle$ be a model, $w \in W$ and $\pi = \varphi_1?;a_1;\varphi_2?;\ldots;\varphi_n?$ a path expression. Let i be such that $0 \le i \le n$, then*

$$\mathcal{M},w \models rm^\pi_i \textit{ iff there is some } P \in \mathcal{P}^{\mathcal{M}}_\pi \textit{ s.t. } P = w_1a_1w_2\ldots w_n,\ w_i = w$$

and for all $w_j \in P$ we have $\mathcal{M},w_j \models \varphi_j$.

Definition 9. *Let $\pi = \varphi_1?; a_1; \varphi_2?; \ldots; \varphi_n?$, $\varphi = \mathsf{rm}(\pi)\theta$ be an $\mathcal{ML}(\mathsf{rm}, \Diamond^{-1})$-formula*[4]*. We define $\mathsf{Tr}(\varphi)$ as the $\mathcal{ML}(\Diamond^{-1})$-formula resulting of repeatedly applying the following reduction axioms to φ (we assume that $\Diamond_a\psi$ is written as $\neg\Box_a\neg\psi$, and similarly for $\Diamond^{-1}$).*

$$
\begin{array}{lll}
(1)\ \mathsf{rm}(\pi)p & \leftrightarrow p, \quad p \in \mathsf{PROP} \\
(2)\ \mathsf{rm}(\pi)\neg\psi & \leftrightarrow \neg\mathsf{rm}(\pi)\psi \\
(3)\ \mathsf{rm}(\pi)(\psi \wedge \psi') & \leftrightarrow (\mathsf{rm}(\pi)\psi \wedge \mathsf{rm}(\pi)\psi') \\
(4)\ \mathsf{rm}(\pi)\Box_a\psi & \leftrightarrow \Box_a\mathsf{rm}(\pi)\psi, \quad \textit{if } a \notin \pi \\
(5)\ \mathsf{rm}(\pi)\Box_a^{-1}\psi & \leftrightarrow \Box_a^{-1}\mathsf{rm}(\pi)\psi, \quad \textit{if } a \notin \pi \\
(6)\ \mathsf{rm}(\pi)\Box_a\psi & \leftrightarrow (\bigwedge_{i\in\{1,\ldots,n-1\ \mid\ a_i=a\}} \neg rm_i^\pi \to \Box_{a_i}\mathsf{rm}(\pi)\psi)\wedge \\
& \quad (\bigwedge_{i\in\{1,\ldots,n-1\ \mid\ a_i=a\}}(rm_i^\pi \to \Box_{a_i}(rm_{i+1}^\pi \vee \mathsf{rm}(\pi)\psi))) \\
(7)\ \mathsf{rm}(\pi)\Box_a^{-1}\varphi & \leftrightarrow (\bigwedge_{i\in\{1,\ldots,n-1\ \mid\ a_i=a\}} \neg rm_i^\pi \to \Box_{a_i}^{-1}\mathsf{rm}(\pi)\psi)\wedge \\
& \quad (\bigwedge_{i\in\{1,\ldots,n-1\ \mid\ a_i=a\}}(rm_i^\pi \to \Box_{a_i}^{-1}(rm_{i-1}^\pi \vee \mathsf{rm}(\pi)\psi))).
\end{array}
$$

Notice that the resulting formula only contains $\Box_a$ and $\Box_a^{-1}$, and does not contain rm. We will prove that the reduction axioms preserves equivalence. The reduction axioms introduced in Definition 9 are justified by the next proposition.

Proposition 4. *Formulas (1) to (7) in Definition 9 are valid.*

The next proposition establishes that we can reduce $\mathcal{ML}(\mathsf{rm}, \Diamond^{-1})$-formulas according to axioms of Definition 9, obtaining an equivalent $\mathcal{ML}(\Diamond^{-1})$-formula. The proof is a direct corollary of Proposition 4.

Proposition 5. *Let $\mathcal{M} = \langle W, R, V\rangle$ a model, $w \in W$ and φ a $\mathcal{ML}(\mathsf{rm}, \Diamond^{-1})$-formula. Then $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models \mathsf{Tr}(\varphi)$.*

The next theorem now follows.

Theorem 3. *The satisfiability problem for $\mathcal{ML}(\mathsf{rm})$ is decidable.*

The reduction axioms that relate $\Box_a$ and $\mathsf{rm}(\pi)$ produce an exponential blow up in the size of the formula. If we consider only path expressions π of size 1, i.e., we consider the fragment $\mathcal{ML}(\mathsf{rm}^-)$, we can avoid the exponential blow up, and prove that the satisfiability problem is PSPACE-complete.

Proposition 6. *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, θ, φ and ψ be $\mathcal{ML}(\mathsf{rm}^-)$-formulas and $a \in \mathsf{AGT}$. Then*

$$\mathcal{M}, w \models \mathsf{rm}(\varphi?; a; \psi?)\Box_a\theta \textit{ iff } \mathcal{M}, w \models \Box_a((\psi \wedge \Diamond^{-1}\varphi) \vee \mathsf{rm}(\varphi?; a; \psi?)\theta).$$

We showed that there is a polynomial translation from $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ into a dynamic epistemic modal logic with action models with both pre-conditions and post-conditions, that preserves satisfiability. In [3], it is proved that the satisfiability problem for dynamic epistemic modal logic with action models with pre-conditions and *without* post-conditions is in NEXPTIME. We can handle post-conditions in NEXPTIME adapting the tableau method of [3][5].

[4] Let $\mathcal{ML}(\mathsf{rm}, \Diamond^{-1})$ be the fragment $\mathcal{ML}(\mathsf{rm})$ extended with the past operator $\Diamond^{-1}$.

[5] A similar result was shown in [12] for public announcement enriched with public assignments which are similar to post-conditions.

Theorem 4. *The satisfiability problem for $\mathcal{AML}^+$ is in* NExpTime.

Then we can state:

Corollary 2. *The satisfiability problem for $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ is in* NExpTime.

As there is a polynomial translation from dynamic epistemic modal logic without post-conditions $\mathcal{AML}$ into $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ that preserves satisfiability, the satisfiability problem of a formula in $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ is NExpTime-hard.

Theorem 5. *The satisfiability problem for $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ is* NExpTime-*complete.*

5 Conclusion

We proposed the dynamic modal logic $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$ which contains copy and remove operators: the copy operator copies an input model, and the remove operator deletes all paths from an input model that are characterized by a given expression. We investigated some model theoretic properties of $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$ such as bisimulations. In order to give an appropriate notion of bisimulation, we need the same conditions as for the $\Diamond^{-1}$ operator, because we need to differentiate states with respect to the paths that traverse them.

We showed that the action model logic $\mathcal{AML}$, one of the best-known dynamic epistemic logics, can be polynomially embedded in the fragment $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ when we consider action models with only Boolean pre-conditions. The restriction to Boolean pre-conditions is certainly a limitation. We consider this to be the first step into a complete understanding of the full language. The embedding simulates every finite action model with a combination of copy and remove operators. As we mentioned, the embedding can be done within $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$ as it only requires single step removals (i.e., only paths of length one are needed). We showed that the copy and one-step removal themselves correspond to particular action models. As a result we obtain a kind of normal form for action models. By decomposing product updates in sequences of copy and remove operators, it would be possible to characterize large syntactic fragments of $\mathcal{AML}$ with interesting complexities for the satisfiability problem.

We demonstrated that the complexity of the satisfiability of the full language $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$ is NExpTime-hard. The upper bound of this satisfiability problem is still open, but we conjecture that it is decidable. We proved that satisfiability for the fragment $\mathcal{ML}(\mathsf{rm}^-)$ is decidable, that it is PSpace-complete for $\mathcal{ML}(\mathsf{cp})$, and that it is NExpTime-complete for $\mathcal{ML}(\mathsf{cp}, \mathsf{rm}^-)$.

As future work, we plan to extend the analysis of $\mathcal{AML}$ via its embedding in $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$. In particular, we will address the general case in which action model pre-conditions can be arbitrary formulas of lower complexity. The main challenge when considering the full language is that when pre-conditions are not Boolean, successive applications of the rm operator are no longer independent of each other, and a more involved mapping into $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$ is required.

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Appendix

Proofs of Section 2

Without loss of generality we assume that all remove operators have the normal form $\mathsf{rm}(\varphi_1?; a_1; \varphi_2?; a_2; \ldots; a_{n-1}; \varphi_n?)\psi$, where $\varphi_i?$ are arbitrary Boolean formulas, and $a_i \in \mathsf{AGT}$ (we can always add $\top?$ and conjunctions to get this normal form). We introduce two lemmas that will be helpful in the proof Theorem 1.

Lemma 1. *Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ be models, $w \in W$, $w' \in W'$, be such that $\mathcal{M}, w \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$, and $\pi{=}\varphi_1?; a_1; \varphi_2?; a_2; \ldots; a_{n-1}; \varphi_n?$. Then, for all $P \in \mathcal{P}^{\mathcal{M}}(\pi)$ such that $P = w_0 a_0 \ldots \mathbf{w} a_i \ldots w_n$, there is some $P' \in \mathcal{P}^{\mathcal{M}'}(\pi)$, with $P' = w'_0 a_0 \ldots \mathbf{w}' a_i \ldots w'_n$ and for all $j \in \{1, \ldots, n\}$ we have $\mathcal{M}, w_j \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'_j$.*

Proof. Given some $P \in \mathcal{P}^{\mathcal{M}}_{\pi}$, we need to find some $P' \in \mathcal{P}^{\mathcal{M}'}_{\pi}$ satisfying the lemma. Let us construct such P'.

Suppose $P = w_0 a_0 \ldots w a_i \ldots w_n$. Notice that we have the subpath $w a_i w_{i+1}$, which means $(w, w_{i+1}) \in R_{a_i}$. Because $\mathcal{M}, w \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$, by (zig) there is some w'_{i+1} such that $(w', w'_{i+1}) \in R'_{a_i}$ and $\mathcal{M}, w_{i+1} \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'_{i+1}$. For this reason, $\mathcal{M}, w_{i+1} \models \psi$ if and only if $\mathcal{M}', w'_{i+1} \models \psi$, for all ψ (in particular φ_{i+1}). Then, w_{i+1} is a good choice in order to construct P'. We can repeat this process to build the subpath $w' a_i w'_{i+1} \ldots w'_n$. In order to choose w_{i-1}, we can proceed in the same way but using (zig^{-1}), and repeating the process until we reach w'_1. Putting all together, we have constructed the right P'.

For the other direction use (zag) and (zag^{-1}). ☐

Lemma 2. *Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ be two models, $w \in W$ and $w' \in W'$. Then $\mathcal{M}, w \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$ implies $\mathcal{M}_{\mathsf{cp}(\bar{p})}, (w, q) \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}'_{\mathsf{cp}(\bar{p})}, (w', q)$.*

Proof. We have to define a bisimulation $Z \subseteq W_{\mathsf{cp}(\bar{p})} \times W'_{\mathsf{cp}(\bar{p})}$. Because we have $\mathcal{M}, w \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$, we define:

$$Z = \{((v, q), (v', q)) \mid (v, q), (v', q) \in W_{\mathsf{cp}(\bar{p})},\ \text{s.t. } \mathcal{M}, v \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', v'\}.$$

(atomic harmony) holds because $(v, q) Z (v', q)$ if and only if v and v' satisfy (atomic harmony) in the original models, and (v, q) and (v', q) are both labeled by the symbol q. For (zig), suppose we have $(v, q) Z (v', q)$ and $((v, q), (u, r)) \in (R_{\mathsf{cp}(\bar{p})})_a$. Then we know $(v, u) \in R_a$. Because $\mathcal{M}, v \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', v'$, by (zig) there is some u' such that $(v', u') \in R'_a$. Hence, we have $((v', q), (u', r)) \in (R'_{\mathsf{cp}(\bar{p})})_a$. (zag) is straightforward. ☐

Then we can state:

Theorem 1 (Invariance under bisimulation.). *For all $\mathcal{ML}(\mathsf{cp}, \mathsf{rm})$-formula φ, we have $\mathcal{M}, w \leftrightarroweq_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$ implies $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.*

Proof. The proof is by structural induction. Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$, such that $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$.

We only prove the inductive cases for rm and cp.

- $\mathbf{rm}(\boldsymbol{\pi})\boldsymbol{\varphi}$: Suppose $\mathcal{M}, w \models \mathsf{rm}(\pi)\varphi$, then $\mathcal{M}_{\mathsf{rm}(\pi)}, w \models \varphi$, where $\mathcal{M}_{\mathsf{rm}(\pi)} = \langle W, R_{\mathsf{rm}(\pi)}, V\rangle$, $R_{\mathsf{rm}(\pi)} = R \setminus \bigcup_{P \in \mathcal{P}^{\mathcal{M}}_{\pi}, a \in \mathsf{AGT}} edges_a(P)$. $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$ by hypothesis, then (by Lemma 1) there is $P \in \mathcal{P}^{\mathcal{M}}_{\pi}$ iff there is $P' \in \mathcal{P}^{\mathcal{M}'}_{\pi}$. Hence $\mathcal{M}_{\mathsf{rm}(\pi)}, w \leftrightarrows_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}'_{\mathsf{rm}(\pi)}, w'$, and by I.H. $\mathcal{M}'_{\mathsf{rm}(\pi)}, w' \models \varphi$. As a result, $\mathcal{M}', w' \models \mathsf{rm}(\pi)\varphi$.
- $\mathbf{cp}(\bar{\boldsymbol{p}}, \boldsymbol{q})\boldsymbol{\varphi}$: Suppose $\mathcal{M}, w \models \mathsf{cp}(\bar{p}, q)\varphi$. Then we have $\mathcal{M}_{\mathsf{cp}(\bar{p})}, (w, q) \models \varphi$. By $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}', w'$ and Lemma 2, we have $\mathcal{M}_{\mathsf{cp}(\bar{p})}, (w, q) \leftrightarrows_{\mathcal{ML}(\mathsf{cp},\mathsf{rm})} \mathcal{M}'_{\mathsf{cp}(\bar{p})}, (w', q)$. By I.H. $\mathcal{M}'_{\mathsf{cp}(\bar{p})}, (w', q) \models \varphi$. Therefore, $\mathcal{M}', w' \models \mathsf{cp}(\bar{p}, q)\varphi$.

□

Proofs of Section 4

Theorem 3. *Deciding if a formula in $\mathcal{ML}(\mathsf{cp})$ is satisfiable is* PSPACE*-complete.*

Proof (Sketch). Adapt the classic tableau-based algorithm for the basic modal logic (see [7]) to manage sequences of propositional symbols which represent possible copies of the model. As for the original algorithm, it takes polynomial time. □

Lemma 3. *Let $\mathcal{M}=\langle W, R, V\rangle$ be a model, $w \in W$ and $\pi = \varphi_1?; a_1; \varphi_2?; \ldots; \varphi_n?$ a path expression. Let i be such that $0 \leq i \leq n$, then*

$$\mathcal{M}, w \models rm^{\pi}_{i} \textit{ iff there is some } P \in \mathcal{P}^{\mathcal{M}}_{\pi} \textit{ s.t. } P = w_1 a_1 w_2 \ldots w_n,\ w_i = w$$

and for all $w_j \in P$ we have $\mathcal{M}, w_j \models \varphi_j$.

Proof. The proof is by induction on the length of π:

- $\boldsymbol{\pi = \varphi_1?}$: $\mathcal{M}, w \models rm^{\pi}_{1}$ if and only if $\mathcal{M}, w \models \varphi_1$ (by definition of rm^{π}_{i}). But $\mathcal{P}^{\mathcal{M}}_{\varphi_1?} = \{v \mid \mathcal{M}, v \models \varphi_1\}$ (all the paths are singletons satisfying φ_1), then $w \in \mathcal{P}^{\mathcal{M}}_{\varphi_1?}$.
- $\boldsymbol{\pi = \varphi_1?; a_1; \varphi_2?; \ldots; \varphi_n?}$: Suppose $\mathcal{M}, w \models rm^{\pi}_{i}$. By definition of rm^{π}, we have $\mathcal{M}, w \models \Diamond^{-1}_{1,i-1} \wedge \varphi_i \wedge \Diamond_{i,n-1}$. Now, we know:
  1. $\mathcal{M}, w \models \varphi_i$.
  2. $\mathcal{M}, w \models \Diamond^{-1}_{1,i-1}$, then by definition of $\Diamond^{-1}_{i,j}$ we have $\mathcal{M}, w \models \Diamond^{-1}_{a_{i-1}}(\Diamond^{-1}_{1,i-2} \wedge \varphi_{i-1})$. By definition of $\models$, there is some $v \in W$ such that $(v, w) \in R_{a_{i-1}}$ and $\mathcal{M}, v \models \Diamond^{-1}_{1,i-2} \wedge \varphi_{i-1}$. Let us define $\pi_1 = \varphi_1?; a_1; \varphi_2?; \ldots; \varphi_{i-1}?$. Then, by definition of rm^{π}_{i}, we have $\mathcal{M}, v \models rm^{\pi_1}_{i-1}$, and by I.H., there is a path $P_1 \in \mathcal{P}^{\mathcal{M}}_{\pi_1}$ such that $P_1 = w_1 a_1 \ldots w_{i-1}$, with $w_{i-1} = v$ and for all $w_j \in P_1$, $\mathcal{M}, w_j \models \varphi_j$ $(0 \leq j \leq i-1)$.
  3. $\mathcal{M}, w \models \Diamond_{i,n-1}$, then by definition of $\Diamond_{i,j}$ we have $\mathcal{M}, w \models \Diamond_{a_i}(\varphi_{i+1} \wedge \Diamond_{i+1,n-1})$. By definition of $\models$, there is some $t \in W$ such that $(w, t) \in R_{a_i}$

and $\mathcal{M}, t \models \varphi_{i+1} \wedge \Diamond_{i+1,n-1}$. Let us define $\pi_2 = \varphi_{i+1}?; a_{i+1}; \ldots; \varphi_n?$. Then, by definition of rm_i^{π}, we have $\mathcal{M}, t \models rm_{i+1}^{\pi_2}$, and by I.H., there is a path $P_2 \in \mathcal{P}_{\pi_2}^{\mathcal{M}}$ such that $P_2 = w_{i+1}a_{i+1}\ldots w_n$, with $w_{i+1} = t$ and for all $w_j \in P_2$, $\mathcal{M}, w_j \models \varphi_j$ $(i+1 \leq j \leq n)$.
Notice that $\pi = \pi_1; a_{i-1}; \varphi_i?; a_i; \pi_2$. It remains to choose $P = P_1 a_{i-1} w_i a_i P_2$ and we have what we wanted.

□

Proposition 4. *Formulas (1) to (7) in Definition 9 are valid.*

Proof. We prove each of them separately:

1. Suppose $\mathcal{M}, w \models \mathsf{rm}(\pi)p$. By definition of $\models$, we have $\mathcal{M}_{\mathsf{rm}(\pi)}, w \models p$. Because $\mathsf{rm}(\pi)$ keeps the same valuation in the updated model, $w \in V(p)$. Then (by $\models$), $\mathcal{M}, w \models p$.

2. Follows from the self-duality of rm, which is trivial given that it is a global operator.

3. Suppose $\mathcal{M}, w \models \mathsf{rm}(\pi)(\psi \wedge \psi')$. Then, by definition of $\models$, $\mathcal{M}_{\mathsf{rm}(\pi)}, w \models (\psi \wedge \psi')$, which means $\mathcal{M}_{\mathsf{rm}(\pi)}, w \models \psi$ and $\mathcal{M}_{\mathsf{rm}(\pi)}, w \models \psi'$. Applying again definition of $\models$, we have $\mathcal{M}, w \models \mathsf{rm}(\pi)\psi$ and $\mathcal{M}, w \models \mathsf{rm}(\pi)\psi'$, iff $\mathcal{M}, w \models \mathsf{rm}(\pi)\psi \wedge \mathsf{rm}(\pi)\psi'$.

4. (5 is straightforward). Suppose $\mathcal{M}, w \models \mathsf{rm}(\pi)\Box_{a_i}\psi$. Applying definition of $\models$ twice, we have that for all v such that $(w, v) \in (R_{\mathsf{rm}(\pi)})_{a_i}$, $\mathcal{M}_{\mathsf{rm}(\pi)}, v \models \psi$. We assume $a_i \notin \pi$, then $(w, v) \in (R_{\mathsf{rm}(\pi)})_{a_i}$ iff $(w, v) \in R_{a_i}$, then we have for all v such that $(w, v) \in R_{a_i}$, $\mathcal{M}_{\mathsf{rm}(\pi)}, v \models \psi$, iff for all v such that $(w, v) \in R_{a_i}$, $\mathcal{M}, v \models \mathsf{rm}(\pi)\psi$. Hence by $\models$, $\mathcal{M}, w \models \Box_{a_i}\mathsf{rm}(\pi)\psi$.

6. (7 is straightforward). Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, $w \in W$, and let $\mathsf{rm}(\pi)\Box_{a_i}\psi$ be an $\mathcal{ML}(\mathsf{rm}, \Diamond^{-1})$-formula with $\pi = \varphi_1?; a_1; \varphi_2?; \ldots; \varphi_n?$, such that $a_i \in \pi$. We want to prove

$$\mathcal{M}, w \models \mathsf{rm}(\pi)\Box_{a_i}\psi \text{ iff } \mathcal{M}, w \models \delta \wedge \delta'$$

where
$$\delta = \textstyle\bigwedge_{k \in \{1,\ldots,n-1 \mid a_k = a_i\}} \neg rm_k^{\pi} \to \Box_{a_k}\mathsf{rm}(\pi)\psi$$
$$\delta' = \textstyle\bigwedge_{k \in \{1,\ldots,n-1 \mid a_k = a_i\}} (rm_k^{\pi} \to \Box_{a_k}(rm_{k+1}^{\pi} \vee \mathsf{rm}(\pi)\psi)).$$

Let us suppose that $\mathcal{M}, w \models \mathsf{rm}(\pi)\Box_{a_i}\psi$. Then, by definition of $\models$, we have that for all $v \in W$ such that $(w, v) \in (R_{\mathsf{rm}(\pi)})_{a_i}$, $\mathcal{M}_{\mathsf{rm}(\pi)}, v \models \psi$. We will check the two conjuncts δ and δ' separately (for the other direction of the iff, we can assume the two conjuncts together and use the same steps):
1. Suppose $\mathcal{M}, w \models \bigwedge_{k \in \{1,\ldots,n-1 \mid a_k = a_i\}} \neg rm_k^{\pi}$. By definition of $\models$, we have $\mathcal{M}, w \not\models \bigvee_{k \in \{1,\ldots,n-1 \mid a_k = a_i\}} rm_k^{\pi}$. It means that there is no $P \in \mathcal{P}_{\pi}^{\mathcal{M}}$ satisfying Lemma 3, such that $w \in P$, hence no deletions have been done traversing w. Then for all $v \in W$, $(w, v) \in R_{a_i}$ iff $(w, v) \in (R_{\mathsf{rm}(\pi)})_{a_i}$. Because we have for all $v \in W$ such that $(w, v) \in (R_{\mathsf{rm}(\pi)})_{a_i}$, $\mathcal{M}_{\mathsf{rm}(\pi)}, v \models \psi$, then for all $v \in W$ such

that $(w, v) \in R_{a_i}$, $\mathcal{M}_{\mathsf{rm}(\pi)}, v \models \psi$. Therefore, we have for all $v \in W$ such that $(w, v) \in R_{a_i}$, $\mathcal{M}, v \models \mathsf{rm}(\pi)\psi$, then (by $\models$) $\mathcal{M}, w \models \Box_{a_i} \mathsf{rm}(\pi)\psi$.
2. Suppose now for some arbitrary k, $\mathcal{M}, w \models rm^{\pi}_{k}$, where $k \in \{1, \ldots, n-1 \mid a_k = a_i\}$. By Lemma 3 it means that there is a path traversing w that has been deleted. We also know $\mathcal{M}_{\mathsf{rm}(\pi)}, w \models \Box_{a_k}\psi$ by assumption and $k = i$, then for all $v \in W$ such that $(w, v) \in (R_{\mathsf{rm}(\pi)})_{a_k}$, $\mathcal{M}_{\mathsf{rm}(\pi)}, v \models \psi$. Then, for all $u \in W$ such that $(w, u) \in R_{a_k}$, either $\mathcal{M}_{\mathsf{rm}(\pi)}, u \models \psi$ or $u \in P$, with $P \in \mathcal{P}^{\mathcal{M}}_{\pi}$, and u is at position $k + 1$ (because w is at position $k = i$), i.e., $\mathcal{M}, u \models rm^{\pi}_{k+1}$ (by Lemma 3). Therefore, $\mathcal{M}, w \models \Box_{a_k}(rm^{\pi}_{k+1} \vee \mathsf{rm}(\pi)\psi)$. $\Box$

Proposition 6. *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, θ be a $\mathcal{ML}(\mathsf{rm}^-)$-formula, φ and ψ be Boolean formulas and $a \in \mathsf{AGT}$. Then*

$$\mathcal{M}, w \models \mathsf{rm}(\varphi?; a; \psi?)\Box_a\theta \textit{ iff } \mathcal{M}, w \models \Box_a((\psi \wedge \Diamond^{-1}\varphi) \vee \mathsf{rm}(\varphi?; a; \psi?)\theta).$$

Proof. Let us suppose that $\mathcal{M}, w \models \mathsf{rm}(\varphi?; a; \psi?)\Box_a\theta$. Then, we have that for all $v \in W$ s.t. $(w, v) \in (R_{\mathsf{rm}(\varphi?;\mathsf{a};\psi?)})_a$, $\mathcal{M}_{\mathsf{rm}(\varphi?;\mathsf{a};\psi?)}, w \models \theta \quad \otimes$. Let u be s.t. $(w, u) \in R_a$, and let suppose $\mathcal{M}, u \models \neg(\psi \wedge \Diamond^{-1}\varphi)$. This means that $(w, u) \in R$ iff $(w, u) \in (R_{\mathsf{rm}(\varphi?;\mathsf{a};\psi?)})_a$. Then (by $\otimes$) $\mathcal{M}_{\mathsf{rm}(\varphi?;\mathsf{a};\psi?)}, u \models \theta$ iff (by $\models$) $\mathcal{M}, u \models \mathsf{rm}(\varphi?; a; \psi?)\theta$, iff $\mathcal{M}, w \models \Box_a(\neg(\psi \wedge \Diamond^{-1}_a\varphi) \to \mathsf{rm}(\varphi?; a; \psi?)\theta)$. $\Box$

Theorem 4. *The satisfiability problem for $\mathcal{AML}^+$ (i.e., action models with postconditions) is in* NExpTime.

Proof (Sketch). In the following σ denotes a symbol for worlds. Σ', Σ'', etc. denote sequences of pointed action models. The symbol ✓ means that the world survives a sequence of pointed action models.

- $\frac{(\sigma\ \Sigma'\ p)}{(\sigma\ \epsilon\ p)}$ is replaced by $\frac{(\sigma\ \Sigma'\ p)}{(\sigma\ \Sigma''\ p)}$ and $\frac{(\sigma\ \Sigma'\ \neg p)}{(\sigma\ \epsilon\ \neg p)}$ is replaced by $\frac{(\sigma\ \Sigma'\ \neg p)}{(\sigma\ \Sigma''\ \neg p)}$ where $\Sigma' = \Sigma''; \Sigma'''$ such that Σ''' is the longest sequence of pointed action models where p is not modified in the preconditions of current actions;
- Add the rules: $\frac{(\sigma\ \Sigma'\ \checkmark)}{(\sigma\ \Sigma'\ p)}$ if the post-condition in the initial action of the last pointed action model in Σ' makes p true and $\frac{(\sigma\ \Sigma'\ \checkmark)}{(\sigma\ \Sigma'\ \neg p)}$ if it makes p false.

The resulting tableau method can still be turned into a non-deterministic algorithm running in exponential time. $\Box$