Interpolation Results for Default Logic Over Modal Logic

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Abstract

Interpolation is an important meta property of a logic. We study interpolation results for a prioritized variant of Default Logic built over the Modal Logic KDA, the normal modal logic K extended with the axiom D for seriality and the universal modality A.

Keywords: Interpolation, Modal Logic, Default Logic

1 Introduction

Default Logics are among the best-known Nonmonotonic Logics. Their origins can be traced back to Reiter's seminal paper 'A Logic for Default Reasoning' [15]. Since then many variants and *addenda* have been proposed to Reiter's original ideas [2]. Default Logics have been thoroughly studied from the point of view of nonmonotonic consequence relations. However, with some exceptions, in particular [1], little attention has been paid to the study of interpolation results for them. And even less to the study of interpolation results for Default Logics built over Modal Logics.

The combination of Default Logics and Modal Logics is particularly interesting when reasoning about description and prescription. This kind of reasoning is prevalent in diverse areas such as Artificial Intelligence, Software Engineering, Legal Argumentation, etc. Typical descriptive statements refer to basic properties of a domain or scenario. Prescriptive statements are regulatory statements characterising ideal behaviours or situations. One main difficulty in dealing with these kinds of statements occurs when the information regarding the domains changes in a way such that the original prescriptions are overridden; or when prescriptions from different sources contradict each other. The tools of Deontic Logics allow for a distinction between descriptive and prescriptive statements, and the violation and fulfilment of prescriptions; and the tools of Default Logics make it possible to effectively reason about overriding prescriptions, and contradictory descriptions or prescriptions. For these reasons, we develop a Default Logic over Deontic Logic, called $\mathscr{D}$KDA.

We resort to Deontic Logics as they provide a strong logical basis for the study of prescriptions (norms). Deontic Logics originate from the pioneer work of von Wright [16] and have been largely defined as particular Modal Logics [6,4]. The most famous is *Standard Deontic Logic* (SDL), i.e., the normal modal logic K extended with the axiom D for seriality [8,3].

In this short paper, we set out to study interpolation results for $\mathscr{D}\mathsf{KDA}$. Interpolation is an important meta property of a logic [10]. First formulated by Craig in [9], in one of its forms, the property states that if $\Phi \vdash \varphi \supset \psi$, then, there is θ s.t. $\Phi \vdash \varphi \supset \theta$, $\Phi \vdash \theta \supset \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$; where $\vdash$ is syntactical consequence in FOL, $\Phi \cup \{\varphi, \theta, \psi\}$ is a set of FOL formulas, and $\mathscr{L}$ is the set of non-logical symbols of a formula. Interpolation results for Modal Logics can be found in [12,5]. As a property, interpolation is worth studying for has direct applications in the area of theorem proving, the analysis and verification of programs, and in synthesis, e.g., in the generation of invariants. Interpolation also have an application in the structuring of specifications, e.g., in [14] it is proven that a form of interpolation is needed in order to compose specifications (the so-called *Modularization Theorem*). Having in mind similar application areas for $\mathscr{D}\mathsf{KDA}$ it seems natural to try and reproduce some interpolation results for this Default Logic. However, because of its non-monotonic nature and the composite structure of its premiss sets, one of the main challenges regarding interpolation results seems to be finding an adequate notion of interpolation. We will discuss some alternatives, taking advantage of interpolation properties of the underlying Modal Logic.

2 The Modal Logic KDA

Let $\mathscr{P}$ be a denumerable set of *proposition symbols*, the set $\mathscr{F}$ of wffs of KDA is determined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \mathsf{A}\varphi,$$

where $p \in \mathscr{P}$. Other Boolean connectives are obtained from $\neg$ and $\wedge$ in the usual way; $\Box\varphi$ is $\neg\Diamond\neg\varphi$; and $\mathsf{E}\varphi$ is $\neg\mathsf{A}\neg\varphi$. The members of $\mathscr{F}$ are *formulas*. Lowercase Roman letters indicate proposition symbols, lowercase Greek letters indicate formulas, and uppercase Greek letters indicate sets of formulas. Let $\varphi \in \mathscr{F}$, the language of φ, notation $\mathscr{L}(\varphi)$, is its set of propositional symbols.

The semantics of KDA is defined in terms of *Kripke models* that are *serial*. A Kripke model $\mathfrak{M}$ is a tuple $\langle W, R, v\rangle$ where: W is a set of elements (or *worlds*); $R \subseteq W \times W$ is the *accessibility relation*; and $v : W \to \wp(\mathscr{P})$ is the *valuation function*. A Kripke model is *serial* if for every $w \in W$, there is $w' \in W$ s.t. wRw'. Henceforth, we assume that all Kripke models are serial.

Let $\mathfrak{M} = \langle W, R, v\rangle$ be a Kripke model, $w \in W$, and $\varphi \in \mathscr{F}$, define the *satisfiability relation* $\mathfrak{M}, w \models \varphi$ according to the rules below.

$$\begin{array}{lll} \mathfrak{M}, w \models p & \text{iff} & p \in v(w) \\ \mathfrak{M}, w \models \Diamond\varphi & \text{iff} & \text{there is } w' \in W \text{ s.t. } wRw' \text{ and } \mathfrak{M}, w' \models \varphi \\ \mathfrak{M}, w \models \mathsf{A}\varphi & \text{iff} & \text{for all } w' \in W,\ \mathfrak{M}, w' \models \varphi. \end{array}$$

We omit Boolean connectives. A Kripke model $\mathfrak{M} = \langle W, R, v\rangle$ *satisfies* a set of formulas Φ *at a world* $w \in W$, notation $\mathfrak{M}, w \models \Phi$, if $\mathfrak{M}, w \models \varphi$ for all $\varphi \in \Phi$. And it *validates* Φ, notation $\mathfrak{M} \models \Phi$, if $\mathfrak{M}, w \models \Phi$ for all $w \in W$.

Two reasonable notions of *modal logical consequence* between sets of formulas (i.e., premisses), and formulas (i.e., their consequences) are: *global* and *local modal logical consequence*, notation $\models^g$ and $\models^l$, respectively. More precisely, $\Phi \models^g \varphi$ if for every $\mathfrak{M}$, if $\mathfrak{M} \models \Phi$, then, $\mathfrak{M} \models \varphi$. And $\Phi \models^l \varphi$ if for every $\mathfrak{M}$ and w in $\mathfrak{M}$, if $\mathfrak{M}, w \models \Phi$, then, $\mathfrak{M}, w \models \varphi$. The global modality A enables us to handle global and local modal logical consequence uniformly, i.e., $\Gamma \models^g \varphi$ iff $\mathsf{A}(\Gamma) \models^l \varphi$, where $\mathsf{A}(\Gamma) = \{\mathsf{A}\gamma \mid \gamma \in \Gamma\}$ (see [11]). For this reason, we define semantic consequence as local modal logical consequence and drop the superscript l. We write $\models \varphi$ if $\emptyset \models \varphi$. If Γ is finite, $\Gamma \models \varphi$ iff $\models (\bigwedge\Gamma) \supset \varphi$.

We are particularly interested in *interpolation*. This property comes in many flavours [12]. Def. 2.1 introduces some commonly found formulations.

Definition 2.1 [Interpolation] The consequence relation $\models$ has the:

AIP *arrow interpolation property* if whenever $\Phi \models \varphi \supset \psi$, there exists θ s.t.: $\Phi \models \varphi \supset \theta$, $\Phi \models \theta \supset \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

TIP *turnstile interpolation property* if whenever $\Phi \cup \{\varphi\} \models \psi$, there is θ s.t.: $\Phi \cup \{\varphi\} \models \theta$, $\Phi \cup \{\theta\} \models \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

SIP *split interpolation property* if whenever $\Phi \cup \{\varphi_1, \varphi_2\} \models \psi$, there is θ s.t.: $\Phi \cup \{\varphi_1\} \models \theta$, $\Phi \cup \{\varphi_2, \theta\} \models \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi_1) \cap (\mathscr{L}(\varphi_2) \cup \mathscr{L}(\psi))$.

The formula θ in AIP, TIP, and SIP is an *interpolant*.

In FOL, AIP, TIP, and SIP are equivalent to each other. In general this may not be the case (depending on both compactness and the deduction theorem). With the local consequence relation, AIP, TIP and SIP are all equivalent. With the global consequence relation this equivalence might not hold. The moral of the story: attention must be paid to the precise formulation of interpolation.

3 Default Logic over KDA

The set $\mathscr{D}$ of *default rules* over $\mathscr{F}$ contains all figures $\pi : \rho \ / \ \chi$ for $\{\pi, \rho, \chi\} \subseteq \mathscr{F}$. The members of $\mathscr{D}$ are *default rules*. The formula π is called *prerequisite* of the default rule, ρ its *justification*, and χ its *consequent*. We single out Δ for sets of default rules and δ for default rules. For $\Delta \subseteq \mathscr{D}$, $\Pi(\Delta) = \{\pi \mid \pi : \rho \ / \ \chi \in \Delta\}$, $P(\Delta)$ and $X(\Delta)$ are defined similarly for justifications and consequents, resp.

The set $\mathscr{P}$ contains all tuples $\langle \Phi, \Delta, \prec \rangle$, where $\Phi \subseteq \mathscr{F}$, $\Delta \subseteq \mathscr{D}$, and $\prec$ is a partial order on Δ. The members of $\mathscr{P}$ are *(default) premiss sets*. We restrict our attention to cases in which Φ and Δ are finite. $\mathscr{P}$ enables a presentation of a consequence relation for default reasoning over KDA and a justification of such a consequence relation in terms of *extensions*. In this respect, there are two options. For $\varphi \in \mathscr{F}$, φ is a *sceptical default consequence* of $\langle \Phi, \Delta, \prec \rangle \in \mathscr{P}$, notation $\langle \Phi, \Delta, \prec \rangle \mathrel{|\!\!\approx^s} \varphi$, if for *every* extension E of $\langle \Phi, \Delta, \prec \rangle$, $E \models \varphi$. Or φ is a *credulous default consequence* of $\langle \Phi, \Delta, \prec \rangle \in \mathscr{P}$, notation $\langle \Phi, \Delta, \prec \rangle \mathrel{|\!\!\approx^c} \varphi$, if for *some* extension E of $\langle \Phi, \Delta, \prec \rangle$, $E \models \varphi$. In any case, an extension may be seen as an interpretation structure of a syntactical kind (i.e., an extension is a premiss set in KDA taking the usual role of a model).

We skip the formal definition of an extension for the sake of brevity and list some of its properties. An extension of $\langle \Phi, \Delta, \prec \rangle$ is a finite subset of formulas including Φ and closed under the application of default rules. The criterion of application of a default rule is that of Łukaszewicz [13]. Default rules are selected for application in the order in which they appear in a linear extension of $\prec$. This defines a priority relation on default rules [7]. This priority relation differs from some standard approaches in that default rules with lower priority are not included in an extension if this depends on default rules with higher priority. A default premiss set always has one extension but it might have more then one. The set of all extensions of a premiss set is indicated by $\mathscr{E}(\Phi, \Delta, \prec)$.

Let $\models\!\approx$ be either $\models\!\approx^s$ or $\models\!\approx^c$, monotonicity for $\models\!\approx$ is: if $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi$, then $\langle \Phi \cup \Phi', \Delta \cup \Delta', (\prec \cup \prec')^* \rangle \models\!\approx \varphi$. The relation $\models\!\approx$ is non-monotonic.

4 Interpolation in $\mathscr{D}$KDA

It is well known that (local) consequence for KDA has AIP (and hence, TIP and SIP). We now discuss how this affects the interpolation property of the non-monotonic consequence relation $\models\!\approx$.

Definition 4.1 The default consequence relation $\models\!\approx$ has the: *arrow interpolation property* (AIP) if whenever $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \psi$, there is θ s.t.: $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \theta$, $\langle \Phi, \Delta, \prec \rangle \models\!\approx \theta \supset \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$. θ is an *interpolant.*

Proposition 4.2 $\models\!\approx^c$ *has the AIP.*

It is not straightforward to prove whether the AIP holds for $\models\!\approx^s$; and if not, whether a weaker form of this property holds. This said, $\models\!\approx^s$ has the following easily established "interpolation" property.

Definition 4.3 The default consequence relation $\models\!\approx$ has the: $\vee\wedge$-*interpolation property* (OAIP) if whenever $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \psi$, there are θ and θ' s.t.: $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \theta$, $\langle \Phi, \Delta, \prec \rangle \models\!\approx \theta' \supset \psi$, and $\mathscr{L}(\{\theta, \theta'\}) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

Proposition 4.4 $\models\!\approx^s$ *has the OAIP.*

Obviously, if $\models\!\approx$ has the AIP, it has the OAIP. What is interesting about Prop. 4.4 is that θ can be taken to be $\bigvee \theta_i$ and θ' to be $\bigwedge \theta_i$, where each θ_i is an interpolant at the level of extensions of the premiss set.

It is not difficult to formulate versions of the turnstile and the split interpolation properties for $\models\!\approx$; see below.

Definition 4.5 The consequence relation $\models\!\approx$ has the:

TIP *turnstile interpolation property* if whenever $\langle \Phi \cup \{\varphi\}, \Delta \rangle \models\!\approx \psi$, there is θ s.t.: $\langle \Phi \cup \{\varphi\}, \Delta \rangle \models\!\approx \theta$, $\langle \Phi \cup \{\theta\}, \Delta \rangle \models\!\approx \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

SIP *split interpolation property* if whenever $\langle \Phi \cup \{\varphi_1, \varphi_2\}, \Delta \rangle \models\!\approx \psi$, there exists θ s.t.: $\langle \Phi \cup \{\varphi_1\}, \Delta \rangle \models\!\approx \theta$, $\langle \Phi \cup \{\varphi_2, \theta\}, \Delta \rangle \models\!\approx \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi_1) \cap (\mathscr{L}(\varphi_2) \cup \mathscr{L}(\psi))$.

There is an interesting challenge to Def. 4.5: *cumulativity.* This property states: if $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi$ and $\langle \Phi, \Delta, \prec \rangle \models\!\approx \psi$, then, $\langle \Phi \cup \{\varphi\}, \Delta, \prec \rangle \models\!\approx \psi$. Cumulativity fails for $\models\!\approx$. Since TIP and SIP accumulate the interpolant as a sentence in the premiss set, cumulativity might hinder interpolation results in these cases.

5 Discussion and Final Remarks

We discussed interpolation properties for $\mathscr{D}$KDA. Because of the non-monotonic nature of $\mathscr{D}$KDA and the composite structure of its premiss sets, some standard ideas are not applicable in this framework. Our preliminary results are mainly concerned with the definition of adequate notions of interpolation for this logic. As a first step, we took advantage of interpolation results of the underlying modal logic to obtain new notions of interpolation for $\mathscr{D}$KDA. There are many open questions, such as studying possible relations between the interpolants θ and θ' in Def. 4.3. Another interesting question concerns the definition of interpolation properties for $\mid\!\approx^{s}$ which do not look into the internal structure of the extensions of a premiss set. It would also be interesting to study the relation between interpolation and the property of cumulativity.

In this paper we focused on $\mathscr{D}$KDA, but the interpolation notions introduced for this particular case, would hopefully be relevant for other default versions of modal logics.

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