First Steps in Updating Knowing How

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Abstract. We investigate dynamic operations acting over a *knowing how* logic. Our approach makes use of a recently introduced semantics for the *knowing how* operator, based on an indistinguishability relation between plans. This semantics is arguably closer to the standard presentation of *knowing that* modalities in classic epistemic logic. Here, we discuss how the semantics enables us to define dynamic modalities representing different ways in which an agent can learn how to achieve a goal. In this regard, we study two types of updates: ontic updates (for which we provide axiomatizations over a particular class of models), and epistemic updates (for which we investigate some semantic properties).

1 Introduction

Over the last years, a new family of epistemic languages for reasoning about *knowing how* assertions [8] have received much attention. Intuitively, an agent knows how to achieve φ given ψ if she has at her disposal a suitable *course of action* guaranteeing that φ will be the case, whenever she is in a situation in which ψ holds. The concept of knowing how is important not only from a philosophical perspective, but also from a computer science point of view. For instance, it can be seen as a formal account for automated planning and strategic reasoning in AI (see, e.g., [2]).

Most traditional approaches for representing *knowing how* rely in connecting logics of *knowing that* with logics of action (see, e.g., [22,18,14]). However, while a combination of operators for *knowing that* and *ability* (e.g., [26]) produces a *de dicto* concept (*"the agent knows she has an action that guarantees the goal"*), a proper notion of *"knowing how to achieve φ"* requires a *de re* clause (*"the agent has an action that she knows guarantees the goal"*; see [15,13] for a discussion). Based on these considerations, [31,32] introduced a new framework based on a *knowing how* binary modality $\mathsf{Kh}(\psi, \varphi)$. At the semantic level, this language is interpreted over relational models — called in this context labeled transition systems (LTSs). In these models, relations describe the actions an agent has at her disposal (in some sense, her *abilities*). Then, $\mathsf{Kh}(\psi, \varphi)$ holds if and only if there is a "proper plan" (a sequence of actions satisfying certain constraints) in the LTS that unerringly leads from every ψ-state only to φ-states.

While variants of this idea have been explored in the literature (see, for instance, [19,20,9,30]), most of them share a fundamental characteristic: relations are interpreted as the agent's available actions; and the abilities of an agent depend *only* on what these actions can achieve. The framework presented in [5] changed this underlying idea by adding a notion of 'indistinguishability' between plans, related to the notion of *strategy indistinguishability* of, e.g., [16,7]. The intuitive idea is, first, that some plans might not be available to the agent. More importantly, she might consider some of them *indistinguishable* from some others. In such cases, having a proper plan σ that leads from any ψ-state to only φ-states is not enough. Instead, the agent also needs for *all her available plans that she cannot distinguish from* σ to satisfy such requirements. As argued in [5], the benefits of these new semantics are threefold. First, it provides an epistemic 'indistinguishability-based' view of an agent's abilities. Second, it enables us to deal with multi-agent scenarios in a more natural way. Third, this new perspective leads to a natural definition of operators that represent dynamic aspects of *knowing how*, more aligned with *dynamic epistemic logic* (DEL; [28]).

This paper focuses on the latter point. We will make use of the indistinguishability-based semantics to investigate some dynamic operators describing changes in the agents' abilities, and hence in their corresponding epistemic states. To the best of our knowledge, this is the first time in which this problem is addressed (except by the brief discussion introduced in [32] about announcements in the context of knowing how). We start by investigating operators that restrict the models based on some sort of *announcement*, in the spirit of [24]. However, as we will see, this kind of updates in the context of knowing how can be seen as ontic updates, rather than epistemic updates. Then, we will exploit the provided semantics in order to define operations that perform actual epistemic updates. In particular, we will discuss how the indistinguishability relation between plans can be refined in order to perform an epistemic change. We consider our work as the first step towards a dynamic epistemic theory over *knowing how* logics.

Outline. The paper is organized as follows. Sec. 2 recalls the syntax, semantics and a complete axiomatization of the multi-agent *knowing how* logic from [5], discussing also a corresponding notion of bisimulation. These notions are useful in the rest of the paper. Then, Sec. 3 is devoted to investigate different dynamic operators for updating knowing how. First, we introduce ontic updates, based on public announcements [24] and arrow updates [17]. We discuss the properties of the operations, and provide reduction axioms. Then, we provide alternatives for epistemic updates, and discuss some of their semantic properties. In Sec. 4 we offer some final remarks and discuss future lines of work.

2 Basic Definitions

Throughout the text, let Prop be a countable set of propositional symbols, Act a denumerable set of action symbols, and Agt a non-empty finite set of agents.

Definition 1. *Formulas of the language* $\mathsf{L}_{\mathsf{Kh}_i}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi),$$

with $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$*. Other Boolean connectives are defined as usual. The formula* $\mathsf{Kh}_i(\psi, \varphi)$ *is read as* "when ψ is the case, the agent i knows how to make φ true"*. Define also* $\mathsf{A}\varphi := \bigvee_{i \in \mathsf{Agt}} \mathsf{Kh}_i(\neg\varphi, \bot)$ *and* $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$*; they will turn out to be the global universal and existential modalities, respectively.*

In [31,32], formulas are interpreted over *labeled transition systems* (LTSs): relational models in which each (basic) relation indicates the source and target of a particular type of action the agent can perform. In the setting introduced in [5], LTSs are extended with a notion of *uncertainty* between plans.

Definition 2 (Actions and plans). *Let* Act^* *be the set of finite sequences over* Act*. Elements of* Act^* *are called* plans*, with* ϵ *being the* empty plan*. Given* $\sigma \in \mathsf{Act}^*$*, let* $|\sigma|$ *be the length of* σ *(note:* $|\epsilon| := 0$*). For* $0 \le k \le |\sigma|$*, the plan* σ_k *is* σ*'s initial segment up to (and including) the* k*th position (with* $\sigma_0 := \epsilon$*). For* $0 < k \le |\sigma|$*, the action* $\sigma[k]$ *is the one in* σ*'s* k*th position.*

Definition 3 (Uncertainty-based LTS). *An* uncertainty-based LTS ($\mathrm{LTS^U}$) *for* Prop*,* Act *and* Agt *is a tuple* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ *where:* W *is a non-empty set of states (called the domain, and denoted by* $\mathrm{D}_{\mathcal{M}}$*);* $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$ *is a collection of binary relations on* W*;* $\mathbb{S} = \{\mathbb{S}_i \subseteq 2^{\mathsf{Act}^*} \setminus \{\emptyset\} \mid i \in \mathsf{Agt}\}$ *assigns to every agent a non-empty collection of pairwise disjoint non-empty sets of plans:* ***(i)*** $\mathbb{S}_i \neq \emptyset$*,* ***(ii)*** $\pi_1, \pi_2 \in \mathbb{S}_i$ *with* $\pi_1 \neq \pi_2$ *implies* $\pi_1 \cap \pi_2 = \emptyset$*, and* ***(iii)*** $\emptyset \notin \mathbb{S}_i$*; and* $\mathrm{V} : \mathrm{W} \to 2^{\mathsf{Prop}}$ *is a labeling function. Given an* $\mathrm{LTS^U}$ $\mathcal{M}$ *and* $w \in \mathrm{D}_{\mathcal{M}}$*, the pair* $(\mathcal{M}, w)$ *(parenthesis usually dropped) is called a* pointed $\mathrm{LTS^U}$.

Intuitively, $\mathrm{P}_i = \bigcup_{\pi \in \mathbb{S}_i} \pi$ is the set of plans that agent i has at her disposal, and each $\pi \in \mathbb{S}_i$ is an indistinguishability class. Note that, as discussed in [5], there is a one-to-one correspondence between each $\mathbb{S}_i$ and an 'indistinguishability relation' $\sim_i \subseteq \mathrm{P}_i \times \mathrm{P}_i$ describing the agent's *uncertainty* over her available plans ($\sigma_1 \sim_i \sigma_2$ iff there is $\pi \in \mathbb{S}_i$ such that $\{\sigma_1, \sigma_2\} \subseteq \pi$). The presentation used here simplifies the definitions that will follow.

Given her uncertainty over Act^*, the abilities of an agent i depend not on what a single plan can achieve, but rather on what a set of them can guarantee.

Definition 4. *Given* $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$ *and* $\sigma \in \mathsf{Act}^*$*, define* $\mathrm{R}_\sigma \subseteq \mathrm{W} \times \mathrm{W}$ *in the standard way. Then, for* $\pi \subseteq \mathsf{Act}^*$ *and* $U \cup \{u\} \subseteq \mathrm{W}$*, define* $\mathrm{R}_\pi := \bigcup_{\sigma \in \pi} \mathrm{R}_\sigma$*,* $\mathrm{R}_\pi(u) := \bigcup_{\sigma \in \pi} \mathrm{R}_\sigma(u)$*, and* $\mathrm{R}_\pi(U) := \bigcup_{u \in U} \mathrm{R}_\pi(u)$*.*

Definition 5 (Strong executability of plans). *Let* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ *be an* $\mathrm{LTS^U}$*, with* $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$*. A* plan $\sigma \in \mathsf{Act}^*$ *is* strongly executable *(SE) at* $u \in \mathrm{W}$ *if and only if* $v \in \mathrm{R}_{\sigma_k}(u)$ *implies* $\mathrm{R}_{\sigma[k+1]}(v) \neq \emptyset$ *for every* $k \in [0 \ldots |\sigma| - 1]$*. We define the set* $\mathrm{SE}^{\mathcal{M}}(\sigma) := \{w \in \mathrm{W} \mid \sigma \text{ is SE at } w\}$*. Then, a* set of plans $\pi \subseteq \mathsf{Act}^*$ *is* strongly executable *at* $u \in \mathrm{W}$ *if and only if* every *plan* $\sigma \in \pi$ *is* strongly executable *at* u*. Hence,* $\mathrm{SE}^{\mathcal{M}}(\pi) = \bigcap_{\sigma \in \pi} \mathrm{SE}^{\mathcal{M}}(\sigma)$ *is the set of the states in* W *where* π *is strongly executable.*

Thus, a plan is strongly executable (at a state) when *all* its partial executions can be completed. Then, a set of plans is strongly executable when *all* its plans are strongly executable. When the model is clear from the context, we will drop the superscript $\mathcal{M}$ and write simply $\mathrm{SE}(\sigma)$ and $\mathrm{SE}(\pi)$.

Now, we have all the ingredients to define the semantics of the logic.

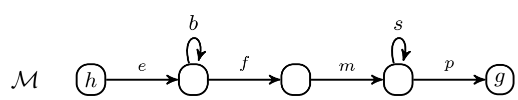
Definition 6. *Let* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V} \rangle$ *be an* $\mathrm{LTS}^{\mathrm{U}}$*; take* $w \in \mathrm{W}$*. The satisfiability relation* $\models$ *for* $\mathsf{L}_{\mathsf{Kh}_i}$ *is inductively defined as:*

$$\begin{array}{lll}
\mathcal{M}, w \models p & \mathit{iff}_{def} & p \in \mathrm{V}(w) \\
\mathcal{M}, w \models \neg\varphi & \mathit{iff}_{def} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \psi \vee \varphi & \mathit{iff}_{def} & \mathcal{M}, w \models \psi \text{ or } \mathcal{M}, w \models \varphi \\
\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi) & \mathit{iff}_{def} & \text{there is } \pi \in \mathbb{S}_i \text{ such that:} \\
 & & \quad \textbf{\textit{(i)}}\ [\![\psi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi), \text{ and} \\
 & & \quad \textbf{\textit{(ii)}}\ \mathrm{R}_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}},
\end{array}$$

where: $[\![\chi]\!]^{\mathcal{M}} := \{w \in \mathrm{W} \mid \mathcal{M}, w \models \chi\}$*. Define:* $\mathcal{M} \models \varphi$ *iff* $[\![\varphi]\!]^{\mathcal{M}} = \mathrm{W}$*, and* $\models \varphi$ *iff* $\mathcal{M} \models \varphi$*, for all* $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$*.*

Note: the above-defined modalities A and E are indeed the global modalities from [11]. Indeed, for every model $\mathcal{M}$ and every state w, $\mathcal{M}, w \models \mathsf{A}\varphi$ holds if and only if φ is true in all states in $\mathcal{M}$ [5].

Example 1. Let us consider a simplified scenario for baking a cake, with two agents i and j. The two agents attempt to produce a good cake (represented by the propositional symbol g). Suppose that they are following a similar recipe, and that they have all the ingredientes (h). The recipe states that g is achieved via the following steps: adding eggs (e), beating the eggs (b), adding flour (f), adding milk (m), stir (s) and finally, bake the preparation (p). Thus, the plan needed to achieve g is *ebfmsp*. Agent i, who is an experienced chef, is aware that is the way to get a good cake. On the other hand, agent j has no cooking experience, so she considers that the order in the instructions do not matter.



$\mathbb{S}_i = \{\,\{\mathit{ebfmsp}\}\,\}$
$\mathbb{S}_j = \{\,\{\mathit{ebfmsp}, \mathit{ebmfsp}\}\,\}$

The diagram shows, on the right, the set of indistinguishable plans in $\mathbb{S}_i$ and in $\mathbb{S}_j$. Notice that agent i knows how to get a good cake, provided that she has all the ingredients (i.e., $\mathcal{M} \models \mathsf{Kh}_i(h, g)$). This is due to the fact that agent i *distinguishes ebfmsp* as the "good plan". On the other hand, as j considers that adding milk and adding flour can be done in any order, we have $\mathcal{M} \not\models \mathsf{Kh}_j(h, g)$.

Bisimulations. Bisimulation is a crucial tool for understanding the expressive power of a formal language. Here we introduce a generalization of the ideas from [10], now for $\mathsf{L}_{\mathsf{Kh}_i}$ over $\mathrm{LTS}^{\mathrm{U}}$s.

Definition 7. *Let* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V} \rangle$ *be an* $\mathrm{LTS}^{\mathrm{U}}$ *over* Prop*,* Act *and* Agt*. Take* $\pi \in 2^{(\mathsf{Act}^*)}$*,* $U, T \subseteq \mathrm{W}$ *and* $i \in \mathsf{Agt}$*.*

- *Write* $U \overset{\pi}{\Rightarrow} T$ iff_{def} $U \subseteq \mathrm{SE}(\pi)$ *and* $\mathrm{R}_\pi(U) \subseteq T$*.*
- *Write* $U \overset{i}{\Rightarrow} T$ iff_{def} *there is* $\pi \in \mathbb{S}_i$ *such that* $U \overset{\pi}{\Rightarrow} T$*.*

Axioms	Taut	$\vdash \varphi$ for φ a propositional tautology
	DistA	$\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$
	TA	$\vdash \mathsf{A}\varphi \to \varphi$
	4KhA	$\vdash \mathsf{Kh}_i(\psi, \varphi) \to \mathsf{A}\mathsf{Kh}_i(\psi, \varphi)$
	5KhA	$\vdash \neg\mathsf{Kh}_i(\psi, \varphi) \to \mathsf{A}\neg\mathsf{Kh}_i(\psi, \varphi)$
	KhE	$\vdash (\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi, \varphi)) \to \mathsf{E}\varphi$
	KhA	$\vdash (\mathsf{A}(\chi \to \psi) \wedge \mathsf{Kh}_i(\psi, \varphi) \wedge \mathsf{A}(\varphi \to \theta)) \to \mathsf{Kh}_i(\chi, \theta)$
Rules	MP	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
	NecA	From $\vdash \varphi$ infer $\vdash \mathsf{A}\varphi$

Table 1: Axiomatization $\mathcal{L}_{\mathsf{Kh}_i}$ for $\mathsf{L}_{\mathsf{Kh}_i}$ w.r.t. $\mathrm{LTS}^{\mathrm{U}}s$.

Additionally, $U \subseteq \mathrm{W}$ is propositionally definable in $\mathcal{M}$ if and only if there is a propositional formula φ such that $U = [\![\varphi]\!]^{\mathcal{M}}$.

Definition 8 ($\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation). *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i\in\mathsf{Agt}}, \mathrm{V}\rangle$ and $\mathcal{M}' = \langle \mathrm{W}', \mathrm{R}', \{\mathbb{S}'_i\}_{i\in\mathsf{Agt}}, \mathrm{V}'\rangle$ be $\mathrm{LTS}^{\mathrm{U}}s$. A non-empty $Z \subseteq \mathrm{W} \times \mathrm{W}'$ is called an $\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if and only if wZw' implies all of the following.*

- ***Atom**: $\mathrm{V}(w) = \mathrm{V}'(w')$.*
- *Kh_i-**Zig**: for any* propositionally *definable $U \subseteq \mathrm{W}$, if $U \overset{i}{\Rightarrow} T$ for some $T \subseteq \mathrm{W}$, then there is $T' \subseteq \mathrm{W}'$ s.t. 1) $Z(U) \overset{i}{\Rightarrow} T'$, and 2) $T' \subseteq Z(T)$.*
- *Kh_i-**Zag**: analogous to Kh_i-**Zig**.*
- *A-**Zig**: for all $u \in \mathrm{W}$ there is a $u' \in \mathrm{W}'$ such that uZu'.*
- *A-**Zag**: for all $u' \in \mathrm{W}'$ there is a $u \in \mathrm{W}$ such that uZu'.*

We write $\mathcal{M}, w \leftrightarrows \mathcal{M}', w'$ when there is an $\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that wZw'.

Theorem 1. *Let $\mathcal{M}, w$ and $\mathcal{M}', w'$ be two $\mathrm{LTS}^{\mathrm{U}}s$. $\mathcal{M}, w \leftrightarrows \mathcal{M}', w'$ implies $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$, for all $\mathsf{L}_{\mathsf{Kh}_i}$-formula φ.*

Axiomatization. We finish this section by recalling an axiom system for $\mathsf{L}_{\mathsf{Kh}_i}$.

Theorem 2 ([5]). *The axiom system from Table 1 is sound and strongly complete w.r.t. the class of all $\mathrm{LTS}^{\mathrm{U}}s$.*

3 Dynamic Knowing How Logics

In this section we will explore different ways in which a *dynamic* operation can be added to $\mathsf{L}_{\mathsf{Kh}_i}$. We can consider a dynamic operator as the indication of performing an update on a model, so that the evaluation of the formula should continue in the modifed model. Some of these model transformations can be interpreted as actions that affect the agents' abilities or her epistemic state. In this section we explore some of these alternatives.

There are at least two ways in which an agent's information might change. It might change because the world changes and she observes this (the *belief update* of the belief change literature; [12]), and it might change because she receives information about the world while the world remains the same (the *belief revision* of the belief change literature; [12]). The former can be called *ontic* change, whereas the latter can be called *epistemic* change. Within dynamic epistemic logic, the first can be represented by a change in valuation, while the second can be represented by changes in the agents' uncertainty [27].

In an $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i\in\mathsf{Agt}}, \mathrm{V}\rangle$, there is a clear distinction between ontic and epistemic information. On the one hand, while R provides *ontic*, *objective* information indicating what the actions themselves can achieve, V describes the actual propositions being true at each state. On the other hand, the *epistemic* state of an agent i (w.r.t. her *knowing how* capabilities) is given by her indistinguishability relation over plans (the set $\mathbb{S}_i$ at her disposal). Hence, in what follows we will consider both ontic and epistemic updates.

3.1 Ontic Updates via Public Announcements

Consider first a model operation removing states (and thus updating the relations). Within the DEL literature, this is interpreted as a *public announcement* (PAL; [24]): an *epistemic* action through which agents get to know publicly that the announced formula is true. Such a *model update* operation is typically described with the operator $[\chi]$, semantically interpreted as

$$\mathcal{M}, w \models [\chi]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \chi \text{ implies } \mathcal{M}_\chi, w \models \varphi,$$

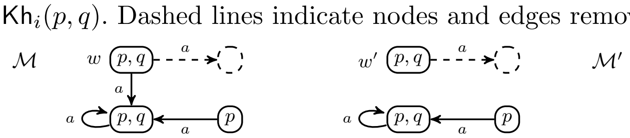
with $\mathcal{M}_\chi$ being the submodel of $\mathcal{M}$ that arises from taking $[\![\chi]\!]^{\mathcal{M}}$ as the new domain, and with the relations and the valuation restricted accordingly (see [28]).

In the original *knowing how* setting from [31], the relations define the agent's abilities. Thus, an update corresponds to both an ontic and an epistemic change (available actions change, and hence so do the agent's abilities). However, in the $\mathrm{LTS}^{\mathrm{U}}$-based semantics, relations provide only ontic information; thus, an update operation produces an *ontic* change, but not an epistemic one.

The update operator adds expressivity to our $\mathsf{L}_{\mathsf{Kh}_i}$ (a similar result was established in [32] for a Kh modality with intermediate constraints).

Proposition 1. *Adding* $[\chi]$ *to* $\mathsf{L}_{\mathsf{Kh}_i}$ *increases its expressive power.*

Proof. The two $\mathrm{LTS}^{\mathrm{U}}$s $\mathcal{M}$ and $\mathcal{M}'$ (with $\mathbb{S}_i = \mathbb{S}'_i = \{\{a\}\}$) below are bisimilar and hence indistinguishable in $\mathsf{L}_{\mathsf{Kh}_i}$. However, $\mathcal{M}, w \models [p]\mathsf{Kh}_i(p,q)$ whereas $\mathcal{M}', w' \not\models [p]\mathsf{Kh}_i(p,q)$. Dashed lines indicate nodes and edges removed after $[p]$.



A consequence of Prop. 1 is that the modality for PAL-like updates is not reducible to the base logic. This makes sense, as the underlying static logic ($\mathsf{L}_{\mathsf{Kh}_i}$) only expresses properties relative to the existence of a way to achieve certain target states from certain origin states. There is no way to characterize the updates produced by $[\chi]$ with the expressive power provided by the Kh_i modality. This is in contrast with what happens when these modalities are added to standard epistemic logic, where reduction axioms can be defined (see, e.g., [28]).

Is it possible to define an alternative, PAL-like update operator, for which reduction axioms exists in $\mathsf{L}_{\mathsf{Kh}_i}$? We will answer this question below.

Definition 9. *Formulas of the language* $\mathsf{PAL}_{\mathsf{Kh}_i}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi,\varphi) \mid [!\varphi]\varphi,$$

with $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$.

Definition 10. *Let* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V}\rangle$ *be an* $\mathrm{LTS}^{\mathrm{U}}$*, and let* χ *be a* $\mathsf{PAL}_{\mathsf{Kh}_i}$*-formula. We define* $\mathcal{M}_{!\chi} = \langle \mathrm{W}_{!\chi}, \mathrm{R}_{!\chi}, \mathbb{S}_{!\chi}, \mathrm{V}_{!\chi}\rangle$*, where:*

- $\mathrm{W}_{!\chi} = [\![\chi]\!]^{\mathcal{M}}$,
- $(\mathrm{R}_{!\chi})_a = \{(w,v) \in \mathrm{R}_a \mid w \in [\![\chi]\!]^{\mathcal{M}},\ \mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}\}$ *for every* $a \in \mathsf{Act}$,
- $\mathbb{S}_{!\chi} = \mathbb{S}$, *and* $\mathrm{V}_{!\chi}(w) = \mathrm{V}(w)$ *(for all* $w \in \mathrm{W}_{!\chi}$*).*

We extend the satisfaction relation $\models$ *from Def. 6 with the case:*

$$\mathcal{M}, w \models [!\chi]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \chi \textit{ implies } \mathcal{M}_{!\chi}, w \models \varphi.$$

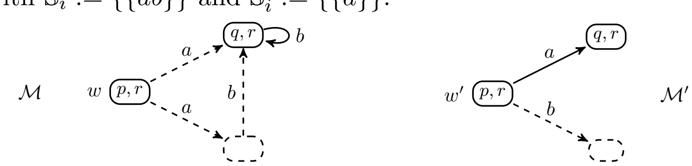
The only difference between the $\mathcal{M}_{!\chi}$ introduced above and the standard $\mathcal{M}_{\chi}$ (which is the restriction of $\mathcal{M}$ to the states satisfying χ) is in the definition of the relations. In the proposal here, a stronger condition is needed for an a-edge from a state $w \in [\![\chi]\!]^{\mathcal{M}}$ to survive after the update: if $\mathrm{R}_a(w) \not\subseteq [\![\chi]\!]^{\mathcal{M}}$ then $(\mathrm{R}_{!\chi})_a(w) = \emptyset$, but if $\mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}$ then $(\mathrm{R}_{!\chi})_a(w) = \mathrm{R}_a(w)$. Notice that in this context, the elimination of some states indicates that the situations they describe are no longer *reachable*, rather than no longer *possible*.

The two forms of model update discussed above bear a resemblance to the two forms of updating neighbourhood models from [21]. Recall that a neighbourhood model [25,23] is given by: a non-empty domain W, an atomic valuation, and a neighbourhood function $\mathrm{N} : \mathrm{W} \to 2^{2^{\mathrm{W}}}$, assigning a set of sets of states to each possible state. Let $U \subseteq \mathrm{W}$ be a non-empty set of states. On the one hand, the U-*intersection* submodel defined in [21] has U as its domain, with its neighbourhood function built by restricting each set in a neighbourhood to the new domain, analogous to what $\mathcal{M}_{\chi}$ (a standard announcement) does. On the other hand, the U-*subset* submodel therein also has U as its domain, but its neighbourhood function is built by keeping only those sets that are already a subset of the new domain, analogous to what $\mathcal{M}_{!\chi}$ does. We argue that this second approach is more appropriate in the context of knowing how.

Even with this, more restricted, version of update, the resulting logic fails to have reduction axioms as the following proposition shows.

Proposition 2. $\mathsf{PAL}_{\mathsf{Kh}_i}$ *is more expresive than* $\mathsf{L}_{\mathsf{Kh}_i}$ *over arbitrary* $\mathrm{LTS}^{\mathrm{U}}$*s.*

Proof. Let $\mathcal{M}$ and $\mathcal{M}'$ be the single agent models depicted below (states and edges depicted with dashed lines are those removed in $\mathcal{M}_{!r}$ and $\mathcal{M}'_{!r}$, respectively), with $\mathbb{S}_i := \{\{ab\}\}$ and $\mathbb{S}'_i := \{\{a\}\}$:



RAtom	$\vdash [!\chi]p \leftrightarrow (\chi \rightarrow p)$
R$\neg$	$\vdash [!\chi]\neg\varphi \leftrightarrow (\chi \rightarrow \neg[!\chi]\varphi)$
R$\vee$	$\vdash [!\chi](\varphi \vee \psi) \leftrightarrow [!\chi]\varphi \vee [!\chi]\psi$
RKh	$\vdash [!\chi]\mathsf{Kh}_i(\varphi, \psi) \leftrightarrow (\chi \rightarrow \mathsf{Kh}_i(\chi \wedge [!\chi]\varphi, \chi \wedge [!\chi]\psi))$
RE$_{[!]}$	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [!\chi]\varphi \leftrightarrow [!\chi]\psi$

Table 2: Reduction axioms $\mathcal{L}_{\mathsf{PAL}_{\mathsf{Kh}_i}}$.

Both models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar (Def. 8); hence, they satisfy the same formulas in $\mathsf{L}_{\mathsf{Kh}_i}$. However, $\mathcal{M}, w \not\models [!r]\mathsf{Kh}_i(p, q)$ since $\mathcal{M}, w \models r$ and $\mathcal{M}_{!r}, w \not\models \mathsf{Kh}_i(p, q)$, whereas $\mathcal{M}', w' \models [!r]\mathsf{Kh}_i(p, q)$ since $\mathcal{M}', w' \models r$ and $\mathcal{M}'_{!r}, w \models \mathsf{Kh}_i(p, q)$.

By furthermore restricting the class of models in which we will evaluate formulas, we are able to obtain reasonable reduction axioms.

Note that $\mathrm{LTS^U}$s contain a set $\mathbb{S}_i$ of sets of plans for each agent i, which determines the perception of the agent with respect to her abilities. For instance, it may be the case that two plans ab and cd belong to some $\pi \in \mathbb{S}_i$, i.e., they are indistinguishable for agent i. In [5] it has been shown that the logic cannot distinguish between the class of arbitrary $\mathrm{LTS^U}$s, and the class of models where each $\pi \in \mathbb{S}_i$ is a singleton with $\pi \subseteq \mathsf{Act}$. This is no longer the case in the presence of $[!\chi]$ (as the proof of Prop. 2 shows).

Definition 11. *Define* $\mathbf{M}^1$ *as the class of models* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ *such that for all* $i \in \mathsf{Agt}$ *and* $\pi \in \mathbb{S}_i$, $\pi \subseteq \mathsf{Act}$.

$\mathbf{M}^1$ constitutes a restricted class of models, which could correspond, for example, to a more abstract representation of the abilities of the agents, in which a course of action is modeled as a single action. The reduction axioms from Table 2 are valid in the class of models $\mathbf{M}^1$. Moreover, we can use them to eliminate announcements by iteratively replacing the innermost occurrence of a $[!\chi]$ modality. Thus, we get completeness for $\mathsf{PAL}_{\mathsf{Kh}_i}$.

Theorem 3. $\mathcal{L}_{\mathsf{Kh}_i}$ *together with the reduction axioms for* $[!\chi]$ *in Table 2 are a sound and strongly complete axiomatization for* $\mathsf{PAL}_{\mathsf{Kh}_i}$ *w.r.t.* $\mathbf{M}^1$.

3.2 Ontic Updates via Arrow Updates

Another framework for modifying relational models is *Arrow Update Logic* (AUL; [17]). It differs from PAL in that it removes only *edges*, thus keeping the domain intact. In standard epistemic logic, this corresponds to changes in uncertainty (e.g., the epistemic indistinguishability might be reduced, so intuitively the agents gain knowledge). For knowing how logics, the situation is different: updating edges in an LTS corresponds to updating the abilities of the agents, as arrows represent execution of actions. We introduce now a logic for arrow updates in the context of our knowing how logic.

Definition 12. *Formulas of the language* $\mathsf{AUL}_{\mathsf{Kh}_i}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi) \mid [U]\varphi,$$
$$U ::= (\varphi, \varphi) \mid U, (\varphi, \varphi),$$

with $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$.

RJoin	$[U]\varphi \leftrightarrow [(\bigwedge_{i=1}^{n} \theta_i, \bigwedge_{i=1}^{n} \theta'_i)]\varphi$
RAtom	$[(\theta,\theta')]p \leftrightarrow p$
R¬	$[(\theta,\theta')]\neg\varphi \leftrightarrow \neg[(\theta,\theta')]\varphi$
R∨	$[(\theta,\theta')](\varphi \vee \psi) \leftrightarrow [(\theta,\theta')]\varphi \vee [(\theta,\theta')]\psi$
RKh	$[(\theta,\theta')]\mathsf{Kh}_i(\varphi,\psi) \leftrightarrow \mathsf{A}([(\theta,\theta')]\varphi \to \theta) \wedge \mathsf{Kh}_i([(\theta,\theta')]\varphi, \theta' \wedge [(\theta,\theta')]\psi)$
RE_U	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [(\theta,\theta')]\varphi \leftrightarrow [(\theta,\theta')]\psi$

Table 3: Reduction axioms $\mathcal{L}_{\mathsf{AUL}_{\mathsf{Kh}_i}}$ with $U = (\theta_1, \theta'_1), \dots, (\theta_n, \theta'_n)$.

Definition 13. *Let* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ *be an* $\mathrm{LTS^U}$*, and* $U = (\theta_1, \theta'_1), \dots, (\theta_n, \theta'_n)$ *be such that* θ_i, θ'_i *are* $\mathsf{AUL}_{\mathsf{Kh}_i}$*-formulas, for all* $0 \le i \le n$*. We define* $\mathcal{M}_U = \langle \mathrm{W}, \mathrm{R}_U, \mathbb{S}, \mathrm{V} \rangle$*, where for every* $a \in \mathsf{Act}$*,*

$$(\mathrm{R}_U)_a = \{(w,v) \in \mathrm{R}_a(w) \mid w \in [\![\textstyle\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}},\ \mathrm{R}_a(w) \subseteq [\![\textstyle\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}\}.$$

Note that if $w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}$, then $\mathrm{R}'_a(w) = \mathrm{R}_a(w)$. Moreover, $\mathrm{R}'_a(w) \neq \emptyset$ iff $w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}}$, $\mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \neq \emptyset$.

Once again, the update here differs from the original one in e.g., [17], in that given a state satisfying the precondition, it takes in consideration all the states that are reachable from it. Thus, the satisfaction of the postcondition at all those states defines whether the arrows are preserved or not.

Definition 14. *We extend the satisfaction relation* $\models$ *from Def. 6 with the case:*

$$\mathcal{M}, w \models [U]\varphi \textit{ iff } \mathcal{M}_U, w \models \varphi.$$

As in the PAL case, AUL performs ontic updates rather than epistemic updates over $\mathrm{LTS^U}$-based knowing how.

Proposition 3. $\mathsf{AUL}_{\mathsf{Kh}_i}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$ *over arbitrary* $\mathrm{LTS^U}$*s.*

Proof. By using the models from Prop. 2, we have that $\mathcal{M}, w \not\models [(r,r)]\mathsf{Kh}_i(p,q)$ and $\mathcal{M}', w' \models [(r,r)]\mathsf{Kh}_i(p,q)$.

Again, the reduction axioms from Table 3 are valid in the class of models $\mathbf{M}^1$, and we can use them to eliminate all the occurrences of the $[U]$ modality.

Theorem 4. $\mathcal{L}_{\mathsf{Kh}_i}$ *together with the reduction axioms for* $[U]$ *in Table 3 are a sound and strongly complete axiomatization for* $\mathsf{AUL}_{\mathsf{Kh}_i}$ *w.r.t.* $\mathbf{M}^1$*.*

3.3 Epistemic Updates, Preliminary Thoughts

In this section we present some preliminary results on different ways in which interesting *epistemic* updates can be introduced in the context of a knowing how operator. No complete axiomatization is available yet. Instead, we will discuss a number of proposals for update operators and show that they can be used to express some relevant properties.

Removing uncertainty between two plans. One of the advantages of $\mathrm{LTS^U}$s is that they allow a natural representation of actions that affect the abilities of an agent, but also her epistemic state. In an $\mathrm{LTS^U}$, the crucial epistemic component is the set $\mathbb{S}_i$, defining not only the plans agent i is 'aware of', but also the level at which she can discern among them. Thus we can represent changes in the epistemic state of an agent by means of operations that modify $\mathbb{S}_i$.

Example 2. Let $\mathcal{M}$ be the LTS$^{\mathrm{U}}$ from Ex. 1. Recall that $\mathcal{M} \not\models \mathsf{Kh}_j(h, g)$. The conflicting plan is *ebmfsp*, which does not lead to a good cake. Thus, if agent j is able to tell apart *ebmfsp* from *ebfmsp* (which is the good plan), she would be able to know how to get a good cake, provided she has the ingredients. If agent j *learns* that the order of the actions matters (so *ebmfsp* is distinct from *ebfmsp*), the set $\pi = \{\mathit{ebfmsp}, \mathit{ebmfsp}\}$ is split into two singleton sets. After such a splitting, she knows how to achieve g given h.

We introduce an operation that eliminates uncertainty between specific plans. In an LTS$^{\mathrm{U}}$, there might be different ways of making distinguishable two previously indistinguishable plans: the different ways one can split a set containing both. First, some notation.

Definition 15. *Let $\pi, \pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$, and $S \subseteq 2^{\mathsf{Act}^*}$. We write $\pi = \pi_1 \uplus \pi_2$ iff $\pi = \pi_1 \cup \pi_2$ and $\pi_1 \cap \pi_2 = \emptyset$.*

For $\pi \in S$ and $\pi = \pi_1 \uplus \pi_2$, define $S^{\pi}_{\{\pi_1,\pi_2\}} \subseteq 2^{\mathsf{Act}^*}$ as the result of refining π through $\{\pi_1, \pi_2\}$: $S^{\pi}_{\{\pi_1,\pi_2\}} := (S \setminus \{\pi\}) \cup \{\pi_1, \pi_2\}$.

Definition 16. *Let $S, S' \subseteq 2^{(\mathsf{Act}^*)}$; and let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$ be such that $\sigma_1 \neq \sigma_2$. We write $S \rightsquigarrow^{\sigma_1}_{\sigma_2} S'$ if and only if either*

- *$S' = S$ and there is no $\pi \in S$ satisfying $\{\sigma_1, \sigma_2\} \subseteq \pi$, or*
- *$S' = S^{\pi}_{\{\pi_1,\pi_2\}}$ for some $\pi \in S$ satisfying $\{\sigma_1, \sigma_2\} \subseteq \pi$, with $\pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$ such that $\pi = \pi_1 \uplus \pi_2$ and $\sigma_1 \in \pi_1$, $\sigma_2 \in \pi_2$.*

Notice that the relation $\rightsquigarrow^{\sigma_1}_{\sigma_2}$ is serial. Moreover, if S is the set of sets of plans for a given agent i in some LTS$^{\mathrm{U}}$ (i.e., $S = \mathbb{S}_i$) and S' is a set satisfying $S \rightsquigarrow^{\sigma_1}_{\sigma_2} S'$, then the structure resulting from replacing S by S' is an LTS$^{\mathrm{U}}$.

Definition 17. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ be an LTS$^{\mathrm{U}}$, and let $\mathbb{S}' = \{\mathbb{S}'_i\}_{i \in \mathsf{Agt}}$ with $\mathbb{S}'_i \subseteq 2^{(\mathsf{Act}^*)}$. Let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$. We write $\mathbb{S} \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}'$ iff for each $i \in \mathsf{Agt}$, $\mathbb{S}_i \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}'_i$. We denote by $\mathcal{M}^{\mathbb{S}}_{\mathbb{S}'}$ the LTS$^{\mathrm{U}}$ obtained by replacing $\mathbb{S}$ by $\mathbb{S}'$.*

The definition above guarantees there is a one-to-one correspondence between the sets in $\mathbb{S}$ and those in $\mathbb{S}'$. With these tools at hand, we introduce the new modality $\langle \sigma_1 \not\sim \sigma_2 \rangle$, semantically interpreted as an action through which all agents learn that plans σ_1 and σ_2 are different. We use $\mathsf{L}_{\mathsf{Ref}}$ (Ref for "refinement") to denote the extension of $\mathsf{L}_{\mathsf{Kh}_i}$ with $\langle \sigma_1 \not\sim \sigma_2 \rangle$.

Definition 18. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ be an LTS$^{\mathrm{U}}$ and $w \in \mathrm{W}$. For $\sigma_1 \neq \sigma_2$,*

$$\mathcal{M}, w \models \langle \sigma_1 \not\sim \sigma_2 \rangle \varphi \quad \mathit{iff}_{\mathit{def}} \quad \text{there is } \mathbb{S}' \text{ s.t. } \mathbb{S} \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}' \text{ and } \mathcal{M}^{\mathbb{S}}_{\mathbb{S}'}, w \models \varphi.$$

As usual, we define $[\sigma_1 \not\sim \sigma_2]\varphi := \neg\langle \sigma_1 \not\sim \sigma_2 \rangle \neg\varphi$.

Formulas of the form $\langle \sigma_1 \not\sim \sigma_2 \rangle \varphi$ can be read as follows: *"after it is stated that plans σ_1 and σ_2 are distinguishable, φ holds"*. For instance, taking Ex. 2, $\langle \mathit{ebmfsp} \not\sim \mathit{ebfmsp} \rangle \mathsf{Kh}_j(h, g)$, establishes that *"after it is stated that ebmfsp and ebfmsp are distinguishable plans, agent j knows how to produce a good cake, provided she has the ingredientes"*.

The proposed modality has some natural properties: it is normal and serial.

Proposition 4. *It follows from the semantics (Def. 18) that:*

1. $\models [\sigma_1 \not\sim \sigma_2](\varphi \to \psi) \to ([\sigma_1 \not\sim \sigma_2]\varphi \to [\sigma_1 \not\sim \sigma_2]\psi)$.
2. *If* $\models \varphi$, *then* $\models [\sigma_1 \not\sim \sigma_2]\varphi$.
3. $\models [\sigma_1 \not\sim \sigma_2]\varphi \to \langle \sigma_1 \not\sim \sigma_2 \rangle\varphi$.

This dynamic modality both preserves knowledge and can generate new one.

Proposition 5. *Let* φ, ψ *be* propositional *formulas. Then,*

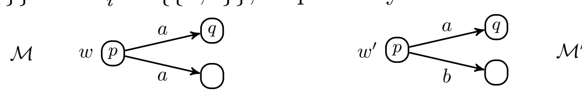
1. $\models \mathsf{Kh}_i(\varphi, \psi) \to [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$.
2. $\neg\mathsf{Kh}_i(\varphi, \psi) \wedge [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$ *is satisfiable.*

Proof. For Item 1, suppose $\mathcal{M}, w \models \mathsf{Kh}_i(\varphi, \psi)$. Then there is $\pi \in \mathbb{S}_i$ s.t. $[\![\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\varphi]\!]^{\mathcal{M}}) \subseteq [\![\psi]\!]^{\mathcal{M}}$. Let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$. If $\sigma_1 \notin \pi$ or $\sigma_2 \notin \pi$, then π does not change and is still the witness for $\mathsf{Kh}_i(\varphi, \psi)$. If, however, $\sigma_1, \sigma_2 \in \pi$, there will be a partition of π, $\{\pi_1, \pi_2\}$ s.t. $\mathbb{S}_i \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}_i{}^{\pi}_{\{\pi_1,\pi_2\}}$. But this does not cause any problem since $[\![\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi) \subseteq \mathrm{SE}(\pi_k)$ and $\mathrm{R}_{\pi_k}([\![\varphi]\!]^{\mathcal{M}}) \subseteq \mathrm{R}_\pi([\![\varphi]\!]^{\mathcal{M}}) \subseteq [\![\psi]\!]^{\mathcal{M}}$, for $k \in \{1, 2\}$. Here agent i knew how to go from φ-states to ψ-states via π. Weakening such π by making a partition still holds the property, allowing the agent to choose between π_1 or π_2 as her next witness. Since all the cases for σ_1 and σ_2 are covered, $\mathcal{M}, w \models [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$. For Item 2, see Ex. 2.

The new modality adds expressivity, as it can talk explicitly about plans:

Proposition 6. $\mathsf{L}_{\mathsf{Ref}}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.

Proof. We need to display two $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar $\mathrm{LTS}^{\mathrm{U}}$s that can be distinguished by an $\mathsf{L}_{\mathsf{Ref}}$-formula. Let $\mathcal{M}$ and $\mathcal{M}'$ be the single agent models depicted below, with $\mathbb{S}_i := \{\{a\}\}$ and $\mathbb{S}'_i := \{\{a, b\}\}$, respectively:



The models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar, thus they satisfy the same formulas in $\mathsf{L}_{\mathsf{Kh}_i}$ (in particular $\neg\mathsf{Kh}_i(p, q)$). But, $\mathcal{M}, w \not\models \langle a \not\sim b \rangle\mathsf{Kh}_i(p, q)$ since $\mathbb{S}_i \rightsquigarrow^a_b \mathbb{S}_i$, whereas $\mathcal{M}', w' \models \langle a \not\sim b \rangle\mathsf{Kh}_i(p, q)$, since there is $\mathbb{S}''_i = \{\{a\}, \{b\}\}$ s.t. $\mathbb{S}'_i \rightsquigarrow^a_b \mathbb{S}''_i$.

Arbitrary refinement over plans. As mentioned, the operation $\langle \sigma_1 \not\sim \sigma_2 \rangle$ can be seen as a particular form of (publicly) removing uncertainty: one indicates precisely the plans that can be distinguished now, and then quantifies over the different ways of doing so. The operation defined below is a more abstract one: in the spirit of other proposals that quantify over epistemic actions (e.g., the arbitrary announcements of [6], the arbitrary arrow updates of [29], the group announcements of [1] and the coalition announcements of [3]), it quantifies over all the different ways in which the agent's indistinguishability can be refined.

Definition 19. *Let* $\mathcal{M}$ *be an* $\mathrm{LTS}^{\mathrm{U}}$ *and* $w \in \mathrm{D}_{\mathcal{M}}$. *Then,*

$$\mathcal{M}, w \models \langle \not\sim \rangle\varphi \;\textit{iff}_{\textit{def}}\; \text{there are } \sigma_1, \sigma_2 \in \mathsf{Act}^* \text{ s.t. } \mathcal{M}, w \models \langle \sigma_1 \not\sim \sigma_2 \rangle\varphi.$$

As usual $[\not\sim]\varphi = \neg\langle \not\sim \rangle\neg\varphi$. *We denote* $\mathsf{L}_{\mathsf{ARef}}$ *(for "arbitrary refinement") as the extension of* $\mathsf{L}_{\mathsf{Kh}_i}$ *with the modality* $\langle \not\sim \rangle$.

The resulting modality is normal and serial, satisfies natural properties of Monotonicity and Weakening, but fails for dynamic versions of axioms 4 and 5.

Proposition 7. *It follows from the semantics (Def. 19) that:*

1. $\models [\not\sim](\varphi \to \psi) \to ([\not\sim]\varphi \to [\not\sim]\psi)$.
2. *If* $\models \varphi$, *then* $\models [\not\sim]\varphi$.
3. $\models [\not\sim]\varphi \to \langle\not\sim\rangle\varphi$.
4. $\models \langle\not\sim\rangle\varphi \to \langle\not\sim\rangle(\varphi \vee \psi)$ *and* $\models [\not\sim]\varphi \to [\not\sim](\varphi \vee \psi)$ *(Monotonicity).*
5. $\models \langle\not\sim\rangle(\varphi \wedge \psi) \to \langle\not\sim\rangle\varphi$ *and* $\models [\not\sim](\varphi \wedge \psi) \to [\not\sim]\varphi$ *(Weakening).*
6. $\not\models [\not\sim]\varphi \to [\not\sim][\not\sim]\varphi$ *(axiom 4).*
7. $\not\models \neg[\not\sim]\varphi \to [\not\sim]\neg[\not\sim]\varphi$ *(axiom 5).*

By definition, $\models \langle\sigma_1 \not\sim \sigma_2\rangle\varphi \to \langle\not\sim\rangle\varphi$, but characterizing the exact expressivity relation between the two resulting logics requires further developments. In particular, given the mismatch between the two languages ($\mathsf{L}_{\mathsf{Ref}}$ is able to talk about specific plans whereas $\mathsf{L}_{\mathsf{ARef}}$ is not), it does not seem trivial to give a translation from one logic to the other. However, by using the same argument as in Prop. 6, it is easy to show the following:

Proposition 8. $\mathsf{L}_{\mathsf{ARef}}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.

Goal directed learning how. One might notice that knowing how operators are *goal-directed*: the agent looks for a suitable course of action that makes her achieve a certain state. It is possible to define an operator that, when possible, *guarantees* that the agent *learns how* to achieve a goal. This action can be understood as a goal-directed learning how: it looks for a way to split *some* existing set of plans π in such a way that the agent knows how to achieve φ given ψ.

Let L_{Lh} (for "learning how") be $\mathsf{L}_{\mathsf{Kh}_i}$ extended with the dynamic modality

$$\langle\psi, \varphi\rangle_i \chi := \langle\not\sim\rangle(\mathsf{Kh}_i(\psi, \varphi) \wedge \chi),$$

(and its 'dual' $[\psi, \varphi]_i \chi := \neg\langle\psi, \varphi\rangle_i \neg\chi$). Moreover, we define $\mathsf{L}_i(\psi, \varphi) := \langle\psi, \varphi\rangle_i \top$ an abbreviation for *"the agent i can learn how to make φ true in the presence of ψ"*. Notice that L_{Lh} is a syntactic fragment of $\mathsf{L}_{\mathsf{ARef}}$.

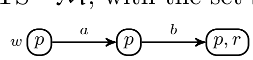
The new dynamic modality is a ternary modality expressing that the agent is able to learn how to achieve φ given ψ, and that after this learning operation takes place, χ holds. The modality L_i is a test of what is learnable by the agent i. The next proposition states some interesting properties of these modalities.

Proposition 9. *It follows from the semantics that:*

1. $\not\models \mathsf{L}_i(\varphi, \psi)$;
2. $\mathsf{L}_i(\varphi, \psi) \wedge \mathsf{L}_i(\varphi, \neg\psi)$ *is satisfiable.*

Proof. Item 1 shows that not everything is learnable by an agent. The (un)availability of certain actions in an $\mathrm{LTS}^{\mathrm{U}}$ restricts what can be learnt. Consider the following single-agent $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$, with the set $\mathbb{S}_i$ shown on the right.

$\mathcal{M}$



$\mathbb{S}_i = \{\, \{ab, a\}, \{\epsilon\} \,\}$

Note that $\mathcal{M}, w \not\models \mathsf{Kh}_i(p, r)$. The set $\{ab, a\}$ is not executable at every p-state, it is only executable at w. On the other hand, $\{\epsilon\}$ is executable everywhere, but does not lead always to r-states. Moreover, $\mathcal{M}, w \not\models \mathsf{L}_i(p, r)$. The set $\{\epsilon\}$ cannot be refined, and no refinement of $\{ab, a\}$ does the work. Therefore, agent i cannot learn how to make r true when p holds.

For Item 2 consider the model $\mathcal{M}'$ in Prop. 6. As said, $\mathcal{M}', w' \not\models \mathsf{Kh}_i(p, q)$. However, there is a way to learn how to achieve q given p: it is possible to split the set $\{a, b\}$ into $\{a\}$ and $\{b\}$; hence, $\mathcal{M}', w' \models \mathsf{L}_i(p, q)$ (witness $\{a\}$) but also $\mathcal{M}', w' \models \mathsf{L}_i(p, \neg q)$ (witness $\{b\}$).

Item 1 shows how, in certain scenarios, there is no room for learning. For instance, there might be no way to learn how to cure a disease, if there is no doctor available. Item 2 shows how the agent might be able to learn not only how to make a formula true under a given condition, but, at the same time, how to make the same formula false under the same condition.

Once more, $[\chi, \psi]$ (seen as a unary modality) is a *normal modality*:

Proposition 10. *The modality* $[\chi, \psi]$ *is normal:*

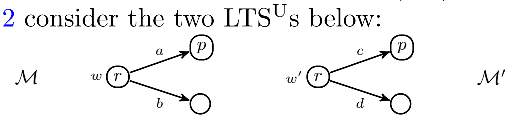
1. $\models [\chi, \psi](\theta \to \varphi) \to ([\chi, \psi]\theta \to [\chi, \psi]\varphi)$.
2. *If* $\models \varphi$, *then* $\models [\chi, \psi]\varphi$.

We finish the section by stating some expressivity connections between the dynamic modalities we just discussed.

Proposition 11. *The following propositions are true:*

1. L_{Lh} *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.
2. L_{Lh} *is not more expressive than* $\mathsf{L}_{\mathsf{Ref}}$.

Proof. Item 1 is proved as Prop. 6: the formula $\mathsf{L}_i(p, q)$ distinghuishes the two LTS^{U}s. For Item 2 consider the two LTS^{U}s below:



For each model, consider respective sets $\mathbb{S}_i = \{\{a, b\}\}$ and $\mathbb{S}'_i = \{\{c, d\}\}$. Since L_{Lh} cannot explicitely talk about plans, $\mathcal{M}, w$ and $\mathcal{M}', w'$ are indistinguishable for it. In $\mathsf{L}_{\mathsf{Ref}}$, $\mathcal{M}, w \models \langle a \not\sim b\rangle \mathsf{Kh}_i(r, p)$ and $\mathcal{M}', w' \not\models \langle a \not\sim b\rangle \mathsf{Kh}_i(r, p)$.

4 Conclusions

Taking the uncertainty-based semantics from [5] as our starting point, we investigated dynamic modalities in the context of *knowing how* logics. In this regard, we studied two forms of updates: ontic updates, via annoucement-like and arrow-update-like modalities; and epistemic updates, refining the perception of an agent regarding her own abilities. For the operators encompassed in the former family, we provided axiomatizations over a particular class of models, via reductions axioms; for the latter family, we discussed some preliminary thoughts and semantic properties of each operator.

We consider this to be the first step towards a more general theory of dynamic epistemic logics for knowing how. Moreover, our work opens the path to study other dynamic operators in this context. For instance, it is known that dynamic operators do not satisfy uniform substitution in general (see, e.g., [4]). It would be interesting to explore alternative techniques for obtaining proof systems without a general rule of substitution. Another approach could be playing with the operators' expressivity (e.g., by expressing other properties about the abilities), in order to find fragments that are axiomatizable via reduction axioms.

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