

# Algebraic Tools for Default Modal Systems

Valentin Cassano<sup>1,2,3,4</sup>

Raul Fervari<sup>1,3,4</sup>

Carlos Areces<sup>1,3</sup>

Pablo F. Castro<sup>2,3</sup>

<sup>1</sup>FAMAF, Universidad Nacional de Córdoba, Argentina

<sup>2</sup>Universidad Nacional de Río Cuarto, Argentina

<sup>3</sup>Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina

<sup>4</sup>Guangdong Technion - Israel Institute of Technology (GTIIT), China

{rfervari, carlos.areces}@unc.edu.ar, {valentin, pcastro}@dc.unrc.edu.ar

## Abstract

Default Logics are a family of non-monotonic formalisms having so-called defaults and extensions as their common foundation. Traditionally, default logics have been defined and dealt with via syntactic notions of consequence in propositional or first-order logic. Here, we build default logics on modal logics. First, we present these default logics syntactically. Then, we elaborate on an algebraic counterpart. More precisely, we extend the notion of a modal algebra to accommodate for defaults and extensions. Our algebraic view of default logics concludes with an algebraic completeness result and a way of comparing default logics borrowing ideas from the concept of bisimulation in modal logic. To our knowledge, this take on default logics approach is novel. Interestingly, it also lays the groundwork for studying default logics from a dynamic logic perspective.

## 1 Introduction

Default Logic refers to a family of non-monotonic formalisms with two main capabilities: reasoning with incomplete knowledge, and dealing with contradictory information. On the one hand, reasoning with incomplete information is handled by so-called defaults. Defaults are non-admissible rules of inference whose conclusions are subject to annulment. Intuitively, defaults handle reasoning with incomplete knowledge by drawing conclusions which complete what is unknown. On the other hand, dealing with contradictory information is handled by so-called extensions. Extensions can be understood as sets of formulas closed under the application of defaults. Intuitively, extensions handle reasoning in the presence of contradictory information by exploring consistent alternatives.

The history of Default Logic traces back to Reiter's seminal work [27]. Since then, many variants of Reiter's original ideas have been proposed. Each variant gives rise to what nowadays can be called a distinct default logic (see [4] for a comprehensive summary). For the most part, these variants have focused their attention on what is meant by an extension. In particular, the emphasis has been on how different interactions between defaults, and the rules of inference of the underlying proof calculus, give rise to different notions of an extension

satisfying one or more properties of interest. In this treatment of extensions, the definition and analysis of a default logic is usually syntactic or proof-theoretic. This leaves the *semantic* side of the coin mostly unexplored. Such a situation calls for revision. Semantic explorations (model theoretic and algebraic alike) yield interesting results: completeness theorems, interpolation properties, model-equivalence relations, algebraic or model invariance properties, etc. Setting them aside on default logics can only be seen as a shortcoming.

**Our work.** We follow the tradition in Default Logic and start with a syntactical formulation of a default logic. We work with default logics defined over modal logics. The definition is carried out resorting to deducibility (i.e., syntactical consequence in a proof calculus). We rely on the notion of global deducibility for modal logics [15]. Our formulation of a default logic is parametric, and can be instantiated with any normal modal system from  $\mathbf{K}$  to  $\mathbf{S5}$  extended with the universal modality [6]. We will see that the use of the universal modality is not arbitrary, but a necessary tool which simplifies the treatment of defaults and extensions in an algebraic setting. For each of our default logics, we make explicit how defaults interact with the rules of inference of the underlying proof calculus. This is done by integrating the use of the former into the notion of deducibility of the latter.

In addition, we explore our construction of a default logic from an algebraic perspective. We do this by extending modal algebras to accommodate for defaults and extensions. Modal algebras are Boolean algebras with additional operators for modalities. They make up the algebraic counterpart of modal systems [35, 17]. We incorporate defaults and extensions into this setting taking Lindenbaum-Tarski constructions as a starting point. Lindenbaum-Tarski constructions act as algebraic canonical models for sets of permisses. We add to these constructions an operator to deal with defaults. This operator can be thought of as “updating” the Lindenbaum-Tarski algebra w.r.t. the application of a default. The result of an update is the algebraic counterpart of an extension. Our algebraic treatment of default logics sets the context to obtain an algebraic completeness result. Moreover, it gives us a way of comparing default logics borrowing ideas from the concept of bisimulation in modal logic.

**Related work.** Our algebraic treatment of defaults and extensions enables us to think of default logics as “model changing” logics. We use the term “model changing” in the sense of, e.g., public announcement logic [26]. In our case, a model change, or update, corresponds to the application of a default (akin to an announcement). The idea of updating a model dynamically to represent syntactic steps while reasoning, can be found in several places in the literature on dynamic logics. For instance, the problem of logical omniscience in epistemic logic (see, e.g., [33]) has been thought of as a property to be achieved after the application of a dynamic operation. In [12, 2, 22, 28], omniscience is achieved by updating models containing sets of formulas. In [32, 20] the updates are performed over awareness relational models. Dynamics of evidence are presented in [31, 34] over neighbourhood models. Finally, dynamic modalities allowing to achieve introspective states over Kripke models are introduced in [13, 14]. Closer to our work is the algebraic treatment of public announcements introduced in [24, 25]. Therein, the algebraic submodel relation induced

by the announcement of a formula  $\psi$  is represented by taking the quotient algebra modulo an equivalence relation given by  $\psi$ . We show that the application of a default  $\delta$  can be thought of in a similar way, i.e., by taking the quotient algebra modulo the equivalence relation given by the new knowledge added by  $\delta$ . We elaborate on this idea in Sec. 4.

**Motivation.** We build default logics over modal logics. Modal logics provide a wide spectrum of formalisms which are both more expressive than propositional logic, and with better computational properties than first-order logic. Modal logics also have a well-developed algebraic theory in terms of modal algebras. In our constructions, we exploit the combination of these two features. It turns out that the effect of applying defaults is better modeled by means of a global consequence relation. We use the universal modality to internalize global consequence into the language. While we do not pursue it here, default logics built on modal logics is interesting from the perspective of applying the developed formalism to particular scenarios. For example, in the setting of description logics, it is possible to think of defaults as a way of capturing exceptions to a taxonomy of concepts modeled in a knowledge base (see [5]).

**Main contributions.** We offer a syntactic and an algebraic treatment of default logics built over modal logics and study their properties. Syntactically, the construction of a default logic over a modal logic results in what we refer to as default modal system. These default systems are parametric on, i.e., determined by, a modal system and a set of defaults. We show how defaults interact with the rules of inference of the underlying modal system by providing a suitable notion of deduction by default. Algebraically, we recast defaults and extensions in the setting of modal algebras. This enables us to obtain a completeness result for default modal systems using standard algebraic tools. Dealing with defaults and extensions in the setting of modal algebras opens the door to study metalogical properties of default modal systems from an algebraic perspective, and can be seen as a first step towards an algebraization of default logics. These contributions extend the ideas and results introduced in [10]. We include examples to illustrate and clarify some important concepts and definitions. We also provide detailed proofs of results. These had been omitted from [10] due to reasons of space.

As a completely novel result, we present a notion of bisimulation for default modal systems. This notion is inspired by the notion of a bisimulation in Modal Logic and enables us to characterize the relative expressivity of a default modal system. We consider this result to be a substantial contribution. Our notion of bisimulation builds on the notion of bisimulation of the underlying modal system. Bisimulations are the de-facto way of proving semantic equivalences in Modal Logic. Herein, we show that bisimulations can also be used to compare default modal systems. Moreover, we elaborate on an application of these results to the problem of equivalence of default theories. Our notion of bisimulation for default modal systems gains in interest since, as it is usual with bisimulations, we only need to inspect properties that are local, i.e., relative to particular points. This is in contrast to other procedures used to compare default theories which are not local but based on complete constructions (see e.g., [30, 21]).

We conclude by discussing how recasting defaults and extensions in the set-

ting of modal algebras lays the groundwork to study default systems from a dynamic logic perspective.

**Structure of the article.** In Sec. 2 we cover some of the background material – Boolean algebras, modal systems, and the algebraization of modal systems. Sec. 3 contains our main results and expands the material in [10] with examples and detailed proofs. More precisely, in Sec. 3.1 we introduce default modal systems. In Sec. 3.2 we present default deducibility. In Sec. 3.3 we provide an algebraic characterization of defaults and extensions. This ultimately yields a completeness theorem. Sec. 3.4 discusses the limitations of our framework. Moreover, we detail what are the challenges and possible directions in generalizing the algebraic treatment of default modal systems. Sec. 3.5 contains novel ideas and results. In particular, in Sec. 3.5, we study some properties of default systems using bisimulations, and discuss an application for comparing so-called default theories. Sec. 4 discusses default modal systems from a dynamic logic perspective. In this section, we relate our algebraic treatment of default modal systems with the algebraic treatment of public announcements introduced in [24, 25]. Sec. 5 offers some final remarks.

## 2 Background

### 2.1 Boolean Algebra in a Nutshell

We briefly introduce the basics of Boolean algebras (see, e.g., [18] for details).

**Definition 2.1.** A *Boolean Algebra* (BA) is a structure  $\mathbf{A} = \langle A, *, -, 1 \rangle$  of type 2–1–0 satisfying a well-known set of equations. The set  $A$  is also denoted as  $|\mathbf{A}|$ . We consider operations  $+$  and  $0$  defined as  $a + b = -(-a * -b)$ , and  $0 = -1$ .

**Definition 2.2.** Every BA  $\mathbf{A}$  is equipped with a partial order  $\preceq_{\mathbf{A}}$  defined as  $x \preceq_{\mathbf{A}} y$  iff  $x = x * y$  (sometimes we omit the subindex  $\mathbf{A}$  and write just  $\preceq$ ). We write  $\uparrow X = \{y \mid \text{there is } x \in X \text{ s.t. } x \preceq y\}$ . A filter is a non-empty subset  $F \subseteq |\mathbf{A}|$  s.t.:  $F = \uparrow F$  and for all  $x, y \in F$ ,  $(x * y) \in F$ . A filter is principal if it is of the form  $\uparrow\{a\}$  for  $a \in |\mathbf{A}|$ . A filter  $F$  is *proper* if  $0 \notin F$ .

### 2.2 Modal Systems

We begin by making precise the set **Form** of well formed formulas we work with.

**Definition 2.3.** Let  $\text{Prop} = \{p_i \mid i \in \mathbb{N}\}$  be a denumerable set of *proposition symbols*; the set **Form** of well formed formulas (wffs, or simply formulas) is determined by the grammar

$$\varphi, \psi ::= p_i \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Box\Box\varphi.$$

We use the usual abbreviations:  $\perp$ ,  $\varphi \vee \psi$ ,  $\varphi \rightarrow \psi$ ,  $\varphi \leftrightarrow \psi$ ,  $\Diamond\varphi$  and  $\Diamond\Box\varphi$ .

The set **Form** can be seen as an enrichment of the basic modal language with the universal modality  $\Box$ . We use the universal modality as a technical tool to internalize a global consequence relation. Def. 2.4 introduces modal systems.



The first item in Prop. 2.1 refers to the *necessitation* property in modal logics, whereas the second item refers to a version of the *deduction theorem*.

We finish this section with a consideration on the the modal systems chosen to work with. Our goal in restricting ourselves to these modal systems was to delimit a well-known class of modal systems on which to build on what we refer to as *default modal systems*. We take these systems to be a good starting point for implementing our ideas on default reasoning and illustrating their reach with a sufficient degree of generality. It remains an open issue to explore whether our ideas apply to other modal systems, or what is the minimal set of properties of a modal system that we necessarily need.

### 2.3 Algebraizing Modal Systems

In this section we present the semantics of a modal system from an algebraic perspective. To this end, we associate with each modal system a class of algebras in a way such that the properties of the modal system are in correspondence to the properties of the class of algebras. The algebraic treatment of modal systems is instrumental to perform default reasoning from a semantic point of view, and to viewing default reasoning as a logic of *updates*. We elaborate on these ideas later on. For now, we focus on introducing some basic concepts and results of the classes of algebras we will work with. To this end, we follow closely [35], and borrow ideas and results from [17, 19].

**Definition 2.6.** The *formula algebra* associated to the set  $\mathbf{Form}$  of formulas is the structure  $\mathbf{F} = \langle \mathbf{Form}, \wedge, \neg, \top, \square, \boxplus \rangle$  where:  $\neg$ ,  $\square$ ,  $\boxplus$  are seen as unary operations, and  $\wedge$  is seen as a binary operation. These operations are defined in the obvious way, i.e.:  $\neg$  applied to  $\varphi \in \mathbf{Form}$  returns  $\neg\varphi \in \mathbf{Form}$ ;  $\square$  applied to  $\varphi \in \mathbf{Form}$  returns  $\square\varphi \in \mathbf{Form}$ ;  $\boxplus$  applied to  $\varphi \in \mathbf{Form}$  returns  $\boxplus\varphi \in \mathbf{Form}$ ; and  $\wedge$  applied to  $\varphi, \psi \in \mathbf{Form}$  returns  $\varphi \wedge \psi \in \mathbf{Form}$ .

The standard algebraic treatment of Classical Propositional Logic uses Boolean algebras (as interpretation structures) and filters (as the semantic counterpart of deducibility). The analogous concepts for the case of modal systems are so-called  $\boxplus$ -modal algebras, and *open filters*, respectively.

**Definition 2.7.** A  $\boxplus$ -modal algebra is a structure  $\mathbf{M} = \langle B, *, -, 1, f^\square, f^\boxplus \rangle$  where:  $\langle B, *, -, 1 \rangle$  is a Boolean algebra; and  $f^\square$  and  $f^\boxplus$  are unary operations satisfying the following set of equations

$$\begin{array}{ll} f^\square(1) = 1 & f^\boxplus(b_1) \preceq b_1 \\ f^\square(b_1 * b_2) = f^\square(b_1) * f^\square(b_2) & f^\boxplus(b_1) \preceq f^\boxplus(-f^\boxplus(-b_1)) \\ f^\boxplus(1) = 1 & f^\boxplus(b_1) \preceq f^\boxplus f^\boxplus(b_1) \\ f^\boxplus(b_1 * b_2) = f^\boxplus(b_1) * f^\boxplus(b_2) & f^\boxplus(b_1) \preceq f^\square(b_1). \end{array}$$

An *open filter* in  $\mathbf{M}$  is a subset  $F \subseteq B$  such that:  $F$  is a filter in  $\langle B, *, -, 1 \rangle$ , and for all  $b \in F$ ,  $f^\boxplus(b) \in F$ .

**Definition 2.8.** An *interpretation* of the formula algebra  $\mathbf{F}$  on a  $\boxplus$ -modal algebra  $\mathbf{M} = \langle B, *, -, 1, f^\square, f^\boxplus \rangle$ , a.k.a. an interpretation on  $\mathbf{M}$ , is a homomorphism  $v : \mathbf{F} \rightarrow \mathbf{M}$  such that:

$$\begin{array}{lll} v(\top) = 1 & v(\neg\varphi) = -v(\varphi) & v(\square\varphi) = f^\square(v(\varphi)) \\ v(\varphi \wedge \psi) = v(\varphi) * v(\psi) & & v(\boxplus\varphi) = f^\boxplus(v(\varphi)). \end{array}$$

**Proposition 2.2.** Every interpretation  $v$  on  $\mathbf{M}$  is uniquely determined by an assignment  $v_0 : \text{Prop} \rightarrow |\mathbf{M}|$ .

**Definition 2.9.** Let  $\mathbf{M}$  be a  $\boxplus$ -modal algebra; we define:

1. an *equation* as a pair  $(\varphi, \psi)$  of formulas; written as  $\varphi \approx \psi$ ;
2. an equation  $\varphi \approx \psi$  is *valid under an interpretation  $v$  on  $\mathbf{M}$*  iff  $v(\varphi) = v(\psi)$ ;
3. an equation is *valid in  $\mathbf{M}$*  iff it is valid under all interpretations on  $\mathbf{M}$ .

We write  $\mathbf{M}, v \models \varphi \approx \psi$  iff the equation  $\varphi \approx \psi$  is valid under  $v$ ; and  $\mathbf{M} \models \varphi \approx \psi$  if the equation  $\varphi \approx \psi$  is valid in  $\mathbf{M}$ .

We are now in a position to connect  $\boxplus$ -modal algebras and modal systems.

**Proposition 2.3.** Let  $\mathbf{M}^\boxplus$  be a modal system, and  $\Phi \cup \{\varphi, \psi\}$  a set of formulas; define  $\varphi \cong_{\mathbf{M}^\boxplus}^\Phi \psi$  iff  $\Phi \vdash_{\mathbf{M}^\boxplus} \varphi \leftrightarrow \psi$ . The relation  $\cong_{\mathbf{M}^\boxplus}^\Phi$  is a congruence on  $\mathbf{F}$ .

**Definition 2.10.** Let  $\mathbf{M}^\boxplus$  be a modal system, and  $\Phi$  be a set of formulas; the  $\mathbf{M}^\boxplus$ -Lindenbaum-Tarski algebra, or  $\mathbf{M}^\boxplus$ -LT algebra, of  $\Phi$  is the structure  $\mathbf{L}_{\mathbf{M}^\boxplus}^\Phi = \langle \text{Form}/\cong_{\mathbf{M}^\boxplus}^\Phi, *_{\cong_{\mathbf{M}^\boxplus}^\Phi}, -_{\cong_{\mathbf{M}^\boxplus}^\Phi}, 1_{\cong_{\mathbf{M}^\boxplus}^\Phi}, f_{\cong_{\mathbf{M}^\boxplus}^\Phi}^\square, f_{\cong_{\mathbf{M}^\boxplus}^\Phi}^\boxplus \rangle$  where:

$$\begin{aligned} 1_{\cong_{\mathbf{M}^\boxplus}^\Phi} &= [\top]_{\cong_{\mathbf{M}^\boxplus}^\Phi} & -_{\cong_{\mathbf{M}^\boxplus}^\Phi}([\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi}) &= [\neg\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi} & f_{\cong_{\mathbf{M}^\boxplus}^\Phi}^\square([\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi}) &= [\square\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi} \\ [\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi} *_{\cong_{\mathbf{M}^\boxplus}^\Phi} [\psi]_{\cong_{\mathbf{M}^\boxplus}^\Phi} &= [\varphi \wedge \psi]_{\cong_{\mathbf{M}^\boxplus}^\Phi} & f_{\cong_{\mathbf{M}^\boxplus}^\Phi}^\boxplus([\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi}) &= [\boxplus\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi}. \end{aligned}$$

The *canonical interpretation*  $v$  on  $\mathbf{L}_{\mathbf{M}^\boxplus}^\Phi$  is defined as  $v(\varphi) = [\varphi]_{\cong_{\mathbf{M}^\boxplus}^\Phi}$ .

**Proposition 2.4.** Every  $\mathbf{M}^\boxplus$ -Lindenbaum-Tarski algebra is a  $\boxplus$ -modal algebra.

**Theorem 2.1.** For every modal system  $\mathbf{M}^\boxplus$ ,  $\Phi \vdash_{\mathbf{M}^\boxplus} \varphi$  iff  $\mathbf{L}_{\mathbf{M}^\boxplus}^\Phi \models \varphi \approx \top$ .

The algebraic completeness of a modal system  $\mathbf{M}^\boxplus$  w.r.t. a corresponding subclass of  $\boxplus$ -modal algebras is obtained as a corollary of Thm. 2.1 by showing that the  $\mathbf{M}^\boxplus$ -LT  $\boxplus$ -modal algebra belongs to the subclass. In this way,  $\mathbf{M}^\boxplus$ -LT  $\boxplus$ -modal algebras act as “algebraic canonical models” for sets of formulas, i.e., they provide witnesses for  $\Phi \not\vdash_{\mathbf{M}^\boxplus} \varphi$ . We make full use of, and benefit from,  $\mathbf{M}^\boxplus$ -LT  $\boxplus$ -modal algebras in Sec. 3.3.

### 3 Default Modal Logic

In this section we integrate the main elements of Default Logic, *defaults* and *extensions*, into modal systems. This integration yields what we call a *default modal system*. For each default modal system, we introduce an associated notion of *deduction by default*, which shows how defaults interact with the notion of deduction for the underlying modal system. Moreover, we present how a default modal system can be viewed from an algebraic perspective, and prove a completeness result using algebraic tools. We also show how the algebraic setting for default modal systems offers a natural way of comparing default logics borrowing ideas from the concept of bisimulation in modal logic. As some final remarks, we discuss how the algebraic treatment of default modal systems can be seen as an update operation on algebraic structures. This opens up the door to thinking about default systems from a dynamic logic perspective (akin to public announcements).

*Remark 1.* To avoid cluttering the notation with subscripts, in what follows, we assume that  $\mathbf{M}^\boxplus$  is an arbitrary but fixed modal system and use  $\vdash$  for  $\vdash_{\mathbf{M}^\boxplus}$ .

### 3.1 Default Modal Systems

We start by introducing defaults and extensions in Defs. 3.1 and 3.2, respectively. These definitions are adapted from [27].

**Definition 3.1.** A default is a triple  $(\pi, \rho, \chi)$  of formulas written as  $\pi : \rho / \chi$ . The formulas  $\pi$ ,  $\rho$ , and  $\chi$ , are called prerequisite, justification, and consequent.

**Definition 3.2.** Let  $\Phi$  be a set of formulas and  $\Delta$  a set of defaults; define a function  $D_{\Delta}^{\Phi}$  s.t. for all sets of formulas  $\Psi$ ,  $D_{\Delta}^{\Phi}(\Psi)$  is the  $\subseteq$ -smallest set of formulas which satisfies:

- (a)  $\Phi \subseteq D_{\Delta}^{\Phi}(\Psi)$ ;
- (b)  $D_{\Delta}^{\Phi}(\Psi) = \{ \psi \mid D_{\Delta}^{\Phi}(\Psi) \vdash_{\mathcal{M}^{\boxplus}} \psi \}$ ;
- (c) for all  $\pi : \rho / \chi \in \Delta$ , if  $\pi \in D_{\Delta}^{\Phi}(\Psi)$  and  $\neg\rho \notin \Psi$ , then,  $\chi \in D_{\Delta}^{\Phi}(\Psi)$ .

A set  $E$  of formulas is an *extension* of  $\Phi$  under  $\Delta$  iff it is a fixed point of  $D_{\Delta}^{\Phi}$ , i.e.,  $E = D_{\Delta}^{\Phi}(E)$ . We use  $E_{\Delta}^{\Phi}$  to indicate the set of all extensions of  $\Phi$  under  $\Delta$ .

In the literature on Default Logic defaults are intuitively understood as defeasible rules of inference, i.e., rules of inference whose conclusions are subject to annulment, or rules which allow us to “jump” to conclusions. In turn, extensions can be thought of akin to theories generated by a set of formulas. In this light, an extension is a set of formulas containing  $\Phi$ , closed under  $\vdash$ , and saturated under the application of the defaults in  $\Delta$ . The next two examples illustrate two properties of extensions: multiplicity and absence of extensions.

*Example 1.* In the context of the modal system  $\mathcal{K}^{\boxplus}$ , consider sets  $\Phi = \{\diamond p\}$  and  $\Delta = \{\diamond p : \diamond\neg p / \diamond\neg p, \diamond p : \Box p / \Box p\}$ ; the set  $E_{\Delta}^{\Phi}$  of extensions of  $\Phi$  under  $\Delta$  consists of exactly two extensions: (1) the set  $E_1 = \{ \varphi \mid \{ \diamond p, \diamond\neg p \} \vdash_{\mathcal{K}^{\boxplus}} \varphi \}$ ; and (2) the set  $E_2 = \{ \varphi \mid \{ \diamond p, \Box p \} \vdash_{\mathcal{K}^{\boxplus}} \varphi \}$ .

Each of the extensions in Ex. 1 corresponds to the application of each default in  $\Delta$ . Once one default has been applied, the application of the other one is blocked. This example illustrates how to handle contradictory information in Default Logic, i.e., via consistent alternatives.

*Example 2.* In the context of the modal system  $\mathcal{K}^{\boxplus}$ , consider sets  $\Phi = \{\diamond p\}$  and  $\Delta = \{\diamond p : \diamond q / \Box\neg q\}$ ; the set  $E_{\Delta}^{\Phi}$  of extensions of  $\Phi$  under  $\Delta$  is empty, i.e.,  $E_{\Delta}^{\Phi} = \emptyset$ , i.e., there are no extensions of  $\Phi$  under  $\Delta$ .

Ex. 2 highlights a subtlety in thinking of extensions as being constructed by the successive application of defaults: applying a default may result in its own annulment. To make this point clear, w.l.o.g., notice that plausible candidates for extensions are: the set  $E_1 = \{ \varphi \mid \{ \diamond p \} \vdash_{\mathcal{K}^{\boxplus}} \varphi \}$  (i.e., not applying the default); or the set  $E_2 = \{ \varphi \mid \{ \diamond p, \Box\neg q \} \vdash_{\mathcal{K}^{\boxplus}} \varphi \}$  (i.e., applying the default). It can easily be verified that neither of these sets is a fixed point of  $D_{\Delta}^{\Phi}$ . More precisely,  $D_{\Delta}^{\Phi}(E_1) = E_2$  and  $D_{\Delta}^{\Phi}(E_2) = E_1$ . This results in  $E_{\Delta}^{\Phi} = \emptyset$ .

The definition of a default modal system arises as a natural construction over a modal system by incorporating defaults and extensions.

**Definition 3.3.** A default modal system is a tuple  $\Delta\mathcal{M}^{\boxplus} = \langle \mathcal{M}^{\boxplus}, \Delta, E \rangle$  where:  $\mathcal{M}^{\boxplus}$  is a modal system,  $\Delta$  is a set of defaults, and  $E$  is a function s.t. for all sets  $\Phi$  of formulas returns  $E_{\Delta}^{\Phi}$ .



In analogy with the case in modal systems, we associate with each default modal system a relation  $\vdash$  between sets of formulas and formulas. This relation is based on the relation  $\vdash$  and it can be understood as its default version. This is made clear in Def. 3.4.

**Definition 3.4.** Let  $\Delta M^\square$  be a default modal system; define

$$\Phi \vdash_{\Delta M^\square} \varphi \quad \text{iff} \quad \varphi \in E \text{ for some } E \in \mathbf{E}_\Delta^\Phi.$$

We drop the subscript  $\Delta M^\square$  when it can be understood from the context. We use  $\vdash \varphi$  as a shorthand for  $\emptyset \vdash \varphi$ .

The relation  $\vdash$  is called *credulous* in the literature on Default Logic. The name is due to the fact that the existence of just one extension is enough to grant the inference (see [4]). It can easily be verified that, in general, *monotonicity* fails for  $\vdash$ . In other words: it may occur that  $\Phi \vdash \varphi$  and  $\Phi \cup \Psi \not\vdash \varphi$ .

Since that  $\vdash$  is built on  $\vdash$ , we may wonder which properties of  $\vdash$  are preserved by  $\vdash$ . There is no obvious answer – notice that, e.g., monotonicity is already not preserved. We introduce Def. 3.5 as a basis on which to start properly frame this question.

**Definition 3.5.** The relation  $\vdash$  interprets  $\vdash$  iff if  $\Phi \vdash \varphi$  then  $\Phi \vdash \varphi$ .

Interpretability seems to be a natural, and perhaps the most basic, requirement on  $\vdash$ . Yet, the example in Ex. 2 (which shows that sometimes extensions do not exist) establishes that this property fails to hold in general. To overcome this problem we can go down two possible paths: (i) modify Def. 3.2 to guarantee the existence of extensions; or (ii) single out defaults for which extensions are guaranteed to exist. For option (i), among the most popular modifications of Def. 3.2 which guarantee the existence of extensions we have: *justified* extensions (see [23]); and *constrained* extensions (see [11]). For option (ii), we have the set of *well-behaved*<sup>1</sup> defaults as a very large and natural set which guarantees the existence of extensions (see [27]). Going down path (i) overburdens the definition of an extension with additional machinery – and takes us away from the purposes of our work here. Instead, we choose to go down path (ii) and restrict ourselves to the case of well-behaved defaults. Interestingly enough, extensions, justified extensions, and constrained extensions, coincide for well-behaved defaults (see [16, 9]).

**Definition 3.6.** A default  $\pi : \rho / \chi$  is well-behaved iff  $\rho = \chi$ . We use  $\pi/\chi$  as notation for well-behaved defaults. A set of defaults  $\Delta$  is well-behaved iff all defaults in  $\Delta$  are well-behaved. A default modal system is well-behaved iff its set of defaults is well-behaved.

**Proposition 3.1.** In every well-behaved default modal system,  $\vdash$  interprets  $\vdash$ .

*Proof.* Notice that  $\Phi \subseteq E$  for all  $E \in \mathbf{E}_\Delta^\Phi$ . The result follows immediately from this and the fact that extensions are guaranteed to exist.

We conclude this section by drawing attention to an interesting point regarding necessitation in default modal systems in Prop. 3.2 (cf. item 1 in Prop. 2.1).

**Proposition 3.2.** In any default modal system, if  $\vdash \varphi$ , then  $\vdash \Box \varphi$ .

<sup>1</sup>In the literature on Default Logic well-behaved defaults are called normal. We avoid using this terminology here to avoid any confusion with normality in Modal Logic.

*Proof.* Suppose that  $\vdash \varphi$ ; by definition, there is an  $E \in E_{\Delta}^{\Phi}$  s.t.  $E \vdash \varphi$ . It follows that  $E \vdash \Box \varphi$ . Thus,  $\vdash \Box \varphi$ .

Prop. 3.2 shows that *necessitation* is preserved by  $\vdash$ . In turn, we may wonder whether it is possible to obtain the form of the deduction theorem in Prop. 2.1 for  $\vdash$ ; i.e., whether if  $\Phi \cup \{\varphi\} \vdash \psi$ , then,  $\Phi \vdash \Box \varphi \rightarrow \psi$ . Unfortunately, as the next example shows, this property fails to hold for an arbitrary default modal systems (even in the presence of  $\Box$ ).

*Example 3.* In the context of the modal system  $K^{\Box}$ , consider sets  $\Phi = \{p\}$  and  $\Delta = \{p/q\}$ ; then,  $E_{\Delta}^{\Phi} = \{\{\varphi \mid \{p, q\} \vdash \varphi\}\}$  and  $E_{\Delta}^{\emptyset} = \{\{\varphi \mid \vdash \varphi\}\}$ . Clearly,  $\{p, q\} \vdash_{K^{\Box}} q$  and  $\not\vdash_{K^{\Box}} \Box p \rightarrow q$ . This means that  $\{p\} \vdash_{\Delta K^{\Box}} q$  and also that  $\not\vdash_{\Delta K^{\Box}} \Box p \rightarrow q$ .

It is worth pointing out that failure of this version of the deduction theorem in a default modal system is caused by the behavior of the use of defaults, and not by the choice of the particular modal system. In other words, the argument used in Ex. 3 replicates itself in all modal systems in the modal cube in Fig. 2.

### 3.2 Deducibility in Default Modal Systems

We formulate a notion of *deduction by default*, or *default deduction*, for an arbitrary but fixed well-behaved default modal system. The notion of deduction by default extends that of deduction by incorporating defaults in a natural way.

**Definition 3.7.** A *deduction by default*, or a *default deduction*, of  $\varphi$  from  $\Phi$  is a finite sequence  $\psi_1 \dots \psi_n$  of formulas s.t.  $\psi_n = \varphi$ , and for each  $k < n$  at least one of the following conditions hold:

1.  $\psi_k$  is a theorem of  $\vdash$ , i.e.,  $\vdash \psi_k$ ;
2.  $\psi_k$  is a premiss, i.e.,  $\psi_k \in \Phi$ ;
3.  $\psi_k$  is obtained using **mp**, i.e., there are  $i, j < k$  s.t.  $\psi_j = \psi_i \rightarrow \psi_k$ ;
4.  $\psi_k$  is obtained using **u**, i.e., there is  $j < k$  s.t.  $\psi_k = \Box \psi_j$ ;
5.  $\psi_k$  is obtained using  $\Delta$ -detachment, i.e., there is  $j < k$  s.t.  $\psi_j / \psi_k \in \Delta$ .

A default deduction is *credulous* whenever:

$$(\Phi \cup \{\psi_i \mid 1 \leq i \leq n\}) \vdash \perp \quad \text{iff} \quad \Phi \vdash \perp. \quad (1)$$

We write  $\Phi \vdash^* \varphi$  iff there is a credulous default deduction of  $\varphi$  from  $\Phi$ .

The notion of a credulous default deduction extends the notion of deduction in the underlying modal system with a rule of default detachment and the condition of being credulous. The rule of default detachment enables us to introduce defaults in the reasoning task and shows us how defaults interact with the rules of the underlying proof system. The condition of being credulous in Eq. (1) captures the fact that defaults cannot be a source of inconsistency. Intuitively, a credulous default deduction of  $\varphi$  from  $\Phi$  internalizes the construction of (part of) an extension containing  $\varphi$  together with the deduction which witnesses this containment. The connection between  $\vdash$  and  $\vdash^*$  is made precise below.

**Theorem 3.1.** For any set of formulas  $\Phi \cup \{\varphi\}$ ,  $\Phi \vdash^* \varphi$  iff  $\Phi \vdash \varphi$ .

*Proof.* W.l.o.g. we prove the result for  $\Phi \not\vdash \perp$ . To simplify the proof, we use an alternative characterization of extensions in terms of *closed generating sequences* (which adapts a definition of a closed process in [3]).

By a  $\Delta$ -sequence we mean a (potentially infinite) sequence  $\bar{\delta} = \delta_1\delta_2\delta_3\dots$  of defaults of  $\Delta$ . The following notation is useful: (a)  $\bar{\delta}|_n = \delta_1\dots\delta_n$ ; (b)  $\delta_i = \pi_i/\chi_i$ ; and (c)  $X_{\bar{\delta}} = \{\chi_i \mid \pi_i/\chi_i \in \bar{\delta}\}$ . A  $\Delta$ -sequence  $\bar{\delta}$  is called *generating* iff for all indices  $i$  of  $\bar{\delta}$ : (d)  $\Phi \cup X_{(\bar{\delta}|_{(i-1)})} \vdash \pi_i$ ; and (e)  $(\Phi \cup X_{(\bar{\delta}|_i)}) \not\vdash \perp$ . A generating  $\Delta$ -sequence is *closed* iff it is not a strict initial segment of any other generating  $\Delta$ -sequence. Extensions and generating  $\Delta$ -sequences are related as follows:  $E \in E_{\Delta}^{\Phi}$  iff exists a generating  $\Delta$ -sequence  $\bar{\delta}$  s.t.  $E = \{\varepsilon \mid (\Phi \cup X_{\bar{\delta}}) \vdash \varepsilon\}$ . The proof of this fact can be obtained by adapting the one presented in [3].

Turning to the proof of Thm. 3.1, we first prove that if  $\Phi \vdash \varphi$ , then  $\Phi \vdash^* \varphi$ . Suppose that for a generating  $\Delta$ -sequence  $\bar{\delta}$ ,  $\Phi \cup X_{\bar{\delta}} \vdash \varphi$ . From compactness for  $\vdash$ , we obtain that for some index  $n$  of  $\bar{\delta}$ ,  $\Phi \cup X_{(\bar{\delta}|_n)} \vdash \varphi$ . We convert a deduction of  $\varphi$  from  $\Phi \cup X_{(\bar{\delta}|_n)}$  into a default deduction of  $\varphi$  from  $\Phi$  in the following way. For each  $\chi_i \in X_{(\bar{\delta}|_n)}$ , there is  $\delta_i = \pi_i/\chi_i \in \bar{\delta}|_n$ ; and so, there is a deduction  $\bar{\psi}_{\pi_i}$  of  $\pi_i$  from  $\Phi \cup X_{(\bar{\delta}|_{(i-1)})}$ . Construct a sequence  $\bar{\psi}_{(\bar{\delta}|_n)} = \bar{\psi}_{\pi_1} \dots \bar{\psi}_{\pi_n}$ . Let  $\bar{\psi}$  be a deduction of  $\varphi$  from  $\Phi \cup X_{(\bar{\delta}|_n)}$ ; the sequence  $\bar{\psi}' = \bar{\psi}_{(\bar{\delta}|_n)}\bar{\psi}$  is a finite sequence of formulas which is, by construction, a default deduction of  $\varphi$  from  $\Phi$ . It can easily be seen that  $\bar{\psi}'$  is also credulous. Thus, if  $\Phi \vdash \varphi$ , then  $\Phi \vdash^* \varphi$ .

To prove that if  $\Phi \vdash^* \varphi$ , then  $\Phi \vdash \varphi$ , we assume that  $\bar{\psi}$  is a credulous default deduction of  $\varphi$  from  $\Phi$ . Let  $\bar{\delta}$  be the  $\Delta$ -sequence of defaults used in  $\bar{\psi}$ , i.e., those collected via default detachment;  $\bar{\delta}$  is, by construction, a generating  $\Delta$ -sequence. It can be proven that  $\bar{\delta}$  can be extended to a generating  $\Delta$ -sequence  $\bar{\delta}'$  that is closed (see [4]). From this fact, the set  $E = \{\varepsilon \mid (\Phi \cup X_{\bar{\delta}'}) \vdash_{M^{\square}} \varepsilon\}$  is an extension. Immediately,  $\varphi \in E$ ; and so  $\Phi \vdash \varphi$ . Thus, if  $\Phi \vdash^* \varphi$ , then  $\Phi \vdash \varphi$ .

In light of Thm. 3.1, we use  $\vdash$  and  $\vdash^*$  interchangeably.

### 3.3 Default Modal Systems Through an Algebraic Lens

We now turn our attention to viewing defaults and extensions in the setting of  $\square$ -modal algebras. We focus on Lindenbaum-Tarski  $\square$ -modal algebras and show how they enable us to think of default modal systems as systems with the ability of performing dynamic updates over a structure.

*Remark 2.* For the rest of this section, we assume  $\Delta M^{\square} = \langle M^{\square}, \Delta, E \rangle$  is an arbitrary but fixed well-behaved default modal system. To simplify notation, we drop  $\Delta M^{\square}$  and  $M^{\square}$  as sub-scripts. Moreover, we write  $\Phi, \varphi$  instead of  $\Phi \cup \{\varphi\}$ .

Def. 3.8 is the cornerstone piece on which we build the rest of this section.

**Definition 3.8.** Let  $\mathcal{L} = \{\mathbf{L}^{\Phi} \mid \Phi \subseteq \text{Form}\}$ ; for every default  $\delta = \pi/\chi \in \Delta$ ; define a function  $\hat{\delta} : \mathcal{L} \rightarrow \mathcal{L}$  s.t.:

$$\hat{\delta}(\mathbf{L}^{\Phi}) = \begin{cases} \mathbf{L}^{\Phi, \chi} & \text{if } [\pi]_{\Phi} = 1_{\Phi} \text{ and } 0_{\Phi} \notin \uparrow\{\llbracket \chi \rrbracket_{\Phi}\} \\ \mathbf{L}^{\Phi} & \text{otherwise.} \end{cases} \quad (2a)$$

$$(2b)$$

The function  $\hat{\delta}$  in Def. 3.8 captures the effect of applying a default from an algebraic perspective. More precisely, applying a default  $\delta = \pi/\chi$  w.r.t. a set  $\Phi$  of formulas yields the set  $\Phi, \chi$  of formulas. The default is applicable iff: (a)  $\Phi \vdash \pi$ ; and (b)  $\Phi, \chi \not\vdash \perp$ . In algebraic terms, we capture the application

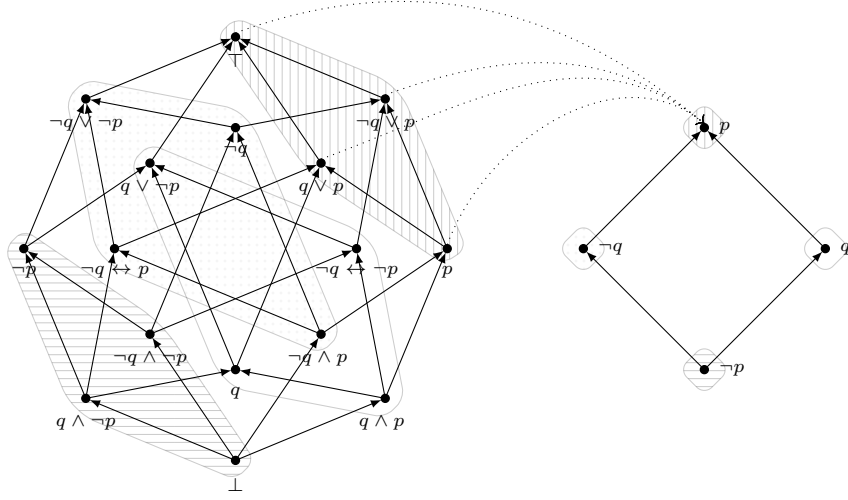


Figure 3: Effect of  $\bar{\delta}$  at the level of Boolean algebras.

of a default as a transformation between LT  $\boxminus$ -modal algebras. Now, consider the LT  $\boxminus$ -modal algebra for a set  $\Phi$  of formulas  $\mathbf{L}^\Phi$ . The condition (a) of applicability of  $\delta = \pi/\chi$  w.r.t.  $\mathbf{L}^\Phi$  is captured in (2a) as  $[\pi]_\Phi = 1_\Phi$ ; and the condition (b) of applicability is captured in (2a) as  $0_\Phi \notin \uparrow\{[\boxminus\chi]_\Phi\}$ . In other words, the equivalence class of  $1_\Phi$  captures the deducibility of  $\pi$  from  $\Phi$ . In turn, the condition of *being proper* on the (open) filter generated by  $[\boxminus\chi]_\Phi$  captures the consistency of  $\chi$  w.r.t.  $\Phi$ . Notice that if the default is applicable, the return value of  $\hat{\delta}$ , i.e.,  $\mathbf{L}^{\Phi, \chi}$ , amounts to incorporating  $\chi$  to  $\Phi$ . In contrast, i.e., if  $\delta$  is not applicable,  $\hat{\delta}$  has no effect on  $\mathbf{L}^\Phi$ . When seen in this light, the function  $\hat{\delta}$  performs an *update* reflecting the application of  $\delta$  on its input. The situation here is similar to the case in logics of updates such as Public Announcement Logic [26] (in particular, in relation to the approach proposed in [24]). We retake this discussion in Sec. 4.

**Proposition 3.3.** Each function  $\hat{\delta}$  induces a function  $\bar{\delta} : |\mathbf{L}| \rightarrow |\hat{\delta}(\mathbf{L})|$  defined as:  $\bar{\delta}([\varphi]_\Phi) = [\varphi]_{\Phi, \chi}$  if Eq. (2a) holds; or  $\bar{\delta}([\varphi]_\Phi) = [\varphi]_\Phi$  if Eq. (2b) holds. The function  $\bar{\delta}$  is a homomorphism from  $\mathbf{L}$  to  $\hat{\delta}(\mathbf{L})$ .

*Proof.* That  $\bar{\delta}$  is a function is trivial. The proof that  $\bar{\delta}$  is a homomorphism is by cases. If Eq. (2b) holds, then, the result is obtained immediately. Otherwise:

$$\bar{\delta}(f_\Phi^\square([\varphi]_\Phi)) = \bar{\delta}([\square\varphi]_\Phi) = [\square\varphi]_{\Phi, \chi} = f_{\Phi, \chi}^\square([\varphi]_{\Phi, \chi}) = f_{\Phi, \chi}^\square(\bar{\delta}([\varphi]_\Phi)).$$

The remaining cases are similar.

In Fig. 3 we explain the effect of applying the function  $\hat{\delta}$  to a LT  $\boxminus$ -modal algebra via  $\bar{\delta}$ . To simplify the drawing, and keep the explanation short and clear, we restrict our attention to what occurs at the level of Boolean algebras. Without further ado, let  $\delta = \top/p$ , and  $\mathbf{L}^\emptyset$  and  $\mathbf{L}^{\{p\}}$  be the LT  $\boxminus$ -modal algebras of  $\emptyset$  and  $\{p\}$ , respectively. The graph on the left represents the Boolean algebra of  $\mathbf{L}^\emptyset$  restricted to language whose proposition symbols include only  $p$  and  $q$ .

In turn, the graph on the right represents the Boolean algebra of  $\mathbf{L}^{\{p\}}$  also restricted to the same propositional symbols. The labels on the nodes of these graphs correspond to equivalence classes whose representatives are the formulas acting as labels. The edges of these graphs correspond to the preorder relation implicit in any Boolean algebra. The area shaded by vertical lines in the graph on the left indicates the elements in the filter generated by  $[p]_{\emptyset}$ . The function  $\bar{\delta}$  maps all the elements in this filter to the top element of the Boolean algebra on the right. This mapping is captured by the dotted arrows. A similar situation occurs with the remaining nodes. The nodes enclosed in each particularly shaded area are mapped by  $\bar{\delta}$  to a corresponding node on the graph on the right. This picture aims to illustrate the kind of “update” that is brought about by a default, i.e., the merging of equivalence classes in the LT  $\boxminus$ -modal algebra.

The following property is immediate.

**Definition 3.9.** Let  $\mathbf{L}_1, \mathbf{L}_2 \in \mathcal{L}$ ; we write  $\mathbf{L}_1 \leq \mathbf{L}_2$  iff there is a homomorphism  $h : \mathbf{L}_1 \rightarrow \mathbf{L}_2$ ; and  $\mathbf{L}_1 < \mathbf{L}_2$  iff  $\mathbf{L}_1 \leq \mathbf{L}_2$  and  $\mathbf{L}_1, \mathbf{L}_2$  are not isomorphic.

**Proposition 3.4.** Every  $\hat{\delta}$  is extensive and idempotent, i.e., it satisfies  $\mathbf{L} \leq \hat{\delta}(\mathbf{L})$  and  $\hat{\delta}(\mathbf{L}) = \hat{\delta}(\hat{\delta}(\mathbf{L}))$ , respectively. An arbitrary  $\hat{\delta}$  needs not satisfy monotonicity, i.e., there are  $\delta = \pi/\chi$  s.t.  $\mathbf{L}_1 \leq \mathbf{L}_2$  and  $\hat{\delta}(\mathbf{L}_1) \not\leq \hat{\delta}(\mathbf{L}_2)$ .

*Proof.* Extensivity follows from Prop. 3.3. Idempotence is proven by cases. If Eq. (2b) holds, then, the result is obtained immediately. Otherwise, Eq. (2a) holds. In this case,  $\hat{\delta}(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi, \chi}$ . Trivially,  $\hat{\delta}(\mathbf{L}^{\Phi, \chi}) = \mathbf{L}^{\Phi, \chi}$ . For a counterexample to monotonicity, consider LT  $\boxminus$ -modal algebras  $\mathbf{L}_{\mathcal{K}^{\boxminus}}^{\emptyset}$  and  $\mathbf{L}_{\mathcal{K}^{\boxminus}}^{\{\square p\}}$  and a default  $\delta = \top/\diamond\neg p$ . Obviously,  $\mathbf{L}_{\mathcal{K}^{\boxminus}}^{\emptyset} \leq \mathbf{L}_{\mathcal{K}^{\boxminus}}^{\{\square p\}}$ . However, there is no homomorphism from  $\hat{\delta}(\mathbf{L}_{\mathcal{K}^{\boxminus}}^{\emptyset})$  to  $\hat{\delta}(\mathbf{L}_{\mathcal{K}^{\boxminus}}^{\{\square p\}})$ .

Having looked at the effect of defaults from an algebraic perspective, we turn our attention to constructing extensions. For well-behaved defaults, extensions can be seen as being constructed in a step-wise fashion applying defaults one at a time. From a syntactic perspective, this construction of an extension starts with a deductively closed set  $\Phi$ , and applies the defaults  $\delta \in \Delta$  one by one until we obtain a deductively closed set  $E \supseteq \Phi$  that is saturated under the application of defaults. From the perspective of LT  $\boxminus$ -modal algebras we obtain that any set  $\Delta$  of well-behaved defaults leads naturally to a set  $\{\hat{\delta} \mid \delta \in \Delta\}$ . Each  $\hat{\delta}$  in this set can be seen as “taking a step” in the construction of the algebraic counterpart of an extension. To carry out this construction in its entirety, we would need to compose such steps. Having to compose steps takes us to Def. 3.10.

**Definition 3.10.** Let  $\mathbf{D}$  be the monoid freely generated by  $\{\hat{\delta} \mid \delta \in \Delta\}$ , i.e.,  $\mathbf{D} = \langle \mathbf{D}, -, -, \text{id} \rangle$  where:

1.  $\mathbf{D}$  is the  $\subseteq$ -smallest set s.t.:
  - (a)  $\{\hat{\delta} : \mathcal{L} \rightarrow \mathcal{L} \mid \delta \in \Delta\} \subseteq \mathbf{D}$ ;
  - (b)  $\text{id} : \mathcal{L} \rightarrow \mathcal{L} \in \mathbf{D}$ ; and
  - (c) if  $\{d_1 : \mathcal{L} \rightarrow \mathcal{L}, d_2 : \mathcal{L} \rightarrow \mathcal{L}\} \subseteq \mathbf{D}$ , then  $(d_1; d_2) : \mathcal{L} \rightarrow \mathcal{L} \in \mathbf{D}$ ;
2.  $\text{id}$  and  $-; -$  satisfy:  $\text{id}(\mathbf{L}) = \mathbf{L}$ ; and  $(d_1; d_2)(\mathbf{L}) = d_2(d_1(\mathbf{L}))$ .

We refer to  $\mathbf{D}$  as the *default monoid* (associated to the Default Modal System).

**Proposition 3.5.** Every  $d \in |\mathbf{D}|$  is either: the identity, i.e.,  $d = \text{id}$ ; or a composition of the form  $d = (\hat{\delta}_1; \dots; \hat{\delta}_n)$ , where  $\delta_i \in \Delta$ .

**Definition 3.11.** Let  $\mathbf{D}$  be a default monoid,  $\mathbf{L}$  be a LT  $\boxminus$ -modal algebra, and  $v$  be an assignment on  $\mathbf{L}$ ; for every equation  $\varphi \approx \psi$ , define:

$$\mathbf{L}, v \vDash \varphi \approx \psi \quad \text{iff} \quad d(\mathbf{L}), (v; \bar{d}) \vDash \varphi \approx \psi \text{ for some } d \in |\mathbf{D}|.$$

where  $\overline{[\varphi]_{\Phi}} = [\varphi]_{\Phi}$ ; and  $\overline{(\hat{\delta}_1; \dots; \hat{\delta}_n)} = (\bar{\delta}_1; \dots; \bar{\delta}_n)$ . We write  $\mathbf{L} \vDash \varphi \approx \psi$  iff  $\mathbf{L}, v \vDash \varphi \approx \psi$  for all assignments  $v$  on  $\mathbf{L}$ . Moreover, we write  $\vDash \varphi \approx \psi$  iff  $\mathbf{L} \vDash \varphi \approx \psi$  for all LT  $\boxminus$ -modal algebras  $\mathbf{L}$ .

Intuitively, the LT  $\boxminus$ -modal algebra  $d(\mathbf{L})$  in Def. 3.11 is the algebraic version of the concept of an extension. This point is made clear in Thm. 3.2.

**Theorem 3.2.** For all sets of formulas  $\Phi, \varphi$ , we have  $\Phi \vdash \varphi$  iff  $\mathbf{L}^{\Phi} \vDash \varphi \approx \top$ .

*Proof.* The interesting part is the right-to-left implication: if  $\mathbf{L}^{\Phi} \vDash \varphi \approx \top$ , then,  $\Phi \vdash \varphi$ . We prove the contrapositive: if  $\Phi \not\vdash \varphi$ , then,  $\mathbf{L}^{\Phi} \not\vDash \varphi \approx \top$ . Let  $\Phi \not\vdash \varphi$ , the proof is concluded if for all  $d \in |\mathbf{D}|$ ,  $d(\mathbf{L}^{\Phi}) \not\vDash \varphi \approx \top$ . We continue by induction on  $d$ .

**Base case:** let  $d = \text{id}$ ; we must have  $\text{id}(\mathbf{L}^{\Phi}) \not\vDash \varphi \approx \top$ ; otherwise we would obtain  $\Phi \vdash \varphi$  (from Thm. 2.1); and so that  $\Phi \vdash \varphi$  (which contradicts our assumption).

**Base case:** let  $d = \hat{\delta}$  for  $\delta = \pi/\chi \in \Delta$ ; either Eq. (2b) holds or Eq. (2a) holds. If Eq. (2b) holds,  $\hat{\delta}$  behaves like  $\text{id}$  (and we are back to the previous case). If Eq. (2a) holds,  $\hat{\delta}(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi, \chi}$ . Assuming (i)  $\mathbf{L}^{\Phi, \chi} \vDash \varphi \approx \top$  leads to a contradiction. More precisely, if Eq. (2a) holds, from Thm. 2.1, we obtain  $\Phi \vdash \pi$  and  $\Phi, \chi \not\vdash \perp$ . From (i) and Thm. 2.1, we obtain  $\Phi, \chi \vdash \varphi$ . If we place the deduction of  $\pi$  from  $\Phi$  in front of the deduction of  $\varphi$  from  $\Phi, \chi$ , we obtain a default deduction of  $\varphi$  from  $\Phi$ . This yields a contradiction.

**Inductive case:** let  $d = (\hat{\delta}_1; \dots; \hat{\delta}_n; \hat{\delta}_{(n+1)})$ . If  $(\hat{\delta}_1; \dots; \hat{\delta}_n)(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi'}$ , using the inductive hypothesis, we get  $\mathbf{L}^{\Phi'} \not\vDash \varphi \approx \top$ . If we assume that  $\hat{\delta}_{(n+1)}(\mathbf{L}^{\Phi'}) \vDash \varphi \approx \psi$ , we get a contradiction using the argument in (i).

### 3.4 Limitations and Generalization of the Framework

It is natural to bring up the question of whether it is possible to free Defs. 3.8, 3.10 and 3.11 from the constraints of LT  $\boxminus$ -modal algebras and to generalize them to arbitrary  $\boxminus$ -modal algebras. We elaborate on some reasons which show that such a generalization is not as immediate as it appears at a first glance. Moreover, we suggest a rather different approach; the details of which are left as further work.

To lay out some context for discussion, we bring attention to the fact that Defs. 3.10 and 3.11 hinge heavily on Def. 3.8. Properly handling the generalization of the function  $\hat{\delta}$  in this last definition to the setting of arbitrary  $\boxminus$ -modal algebras is then the crucial step to take. In what follows we take a closer look at the generalization of this function.

To begin with, a possible way to directly rework the function  $\hat{\delta}$  in Def. 3.8 to encompass the entire class of  $\sqcup$ -modal algebras of a modal system, and not just the class of LT  $\sqcup$ -modal algebras, would be to define it as a function

$$\check{\delta} : \mathcal{M} \rightarrow \mathcal{M},$$

where  $\mathcal{M}$  is the class of  $\sqcup$ -modal algebras of the modal system. At this point, we encounter an obstacle which overburdens the definition of  $\check{\delta}$ . Notice that the canonical interpretation of a formula in a LT  $\sqcup$ -modal algebra is hardcoded in the function  $\hat{\delta}$  of Def. 3.8 for each one of its possible inputs. Taking the same standpoint for the general case implies having, not a single function, but a set of functions  $\check{\delta}$  indexed by sets  $V$  of interpretations. More precisely, it implies having a function  $\check{\delta}_V$  for each set  $V$  s.t.: for every  $\mathbf{M} \in \mathcal{M}$  there is one and only one  $v \in V$  that is an interpretation on  $\mathbf{M}$ . This situation is not satisfactory. It creates a scenario in which we now have to deal with interpretations of formulas in the  $\sqcup$ -modal algebra and “interpretations” of defaults into functions  $\check{\delta}_V$ . It is not difficult to see that going down this path creates a problem with composition in Def. 3.10 and with the quantification of interpretations down the line in Def. 3.11.

Yet, it is clear that we cannot do it without interpretations in the definition of  $\check{\delta}$ . They are, at a minimum, necessary to check the side condition of the function that is being defined. Namely, let  $\delta = \pi/\chi$ , an interpretation  $v$  on  $\mathbf{M}$  is needed to determine the result of  $\check{\delta}(\mathbf{M})$ ; since such a result depends on whether  $v(\pi)$  equals the top element of  $\mathbf{M}$ , and whether the bottom element of  $\mathbf{M}$  belongs to the open filter generated by  $v(\chi)$ . Of course, we could argue that this problem is originated because  $\check{\delta}$  needs to deal with the formulas  $\pi$  and  $\chi$  and a  $\sqcup$ -modal algebra  $\mathbf{M}$ . What if we separate ourselves from the formulas  $\pi$  and  $\chi$  and consider them to be abstract elements? This is no strange situation in algebraic logic, after all, the elements in a  $\sqcup$ -modal algebra can be seen as abstract counterpart of formulas. Suppose then that we define not a function  $\check{\delta}$  but a function

$$f_{(\mathbf{a}/\mathbf{b})} : \mathcal{M} \rightarrow \mathcal{M},$$

for abstract elements  $\mathbf{a}$  and  $\mathbf{b}$ . Intuitively, we would understand  $\mathbf{a}$  and  $\mathbf{b}$  as the abstract counterpart of formulas  $\pi$  and  $\chi$  in a default  $\delta = \pi/\chi$ , respectively. The problem with this standpoint is: What shall we take as  $\mathbf{a}$  and  $\mathbf{b}$ ? Observe that to determine the result of  $f_{(\mathbf{a}/\mathbf{b})}(\mathbf{M})$ , we need to know, e.g., if  $\mathbf{a}$  equals the top element of  $\mathbf{M}$ . This would mean that  $\mathbf{a}$  should be in the universe of  $\mathbf{M}$ . At the same time, for different  $\mathbf{M}'$ , to determine the result of  $f_{(\mathbf{a}/\mathbf{b})}(\mathbf{M}')$ , we need to know whether  $\mathbf{a}$  equals the top element of  $\mathbf{M}'$ . This would mean that  $\mathbf{a}$  should be in the universe of  $\mathbf{M}'$ . But  $\mathbf{M}$  and  $\mathbf{M}'$  are potentially defined over different universes. Thus, we are forced to consider a class of  $\sqcup$ -modal algebras all defined over the same universe; or to consider some notion of correspondence between the universes of the  $\sqcup$ -modal algebras in  $\mathcal{M}$ . This standpoint carries with it another problem which further complicates the problem at hand: We need to consider a way of linking  $\pi$  to  $\mathbf{a}$ , and  $\chi$  to  $\mathbf{b}$ , and this linking should be done in a way such that is coherent with the behavior of  $\mathbf{M}$ . Perhaps more importantly, and the last straw, even if we were capable of solving the problems presented by  $\mathbf{a}$  and  $\mathbf{b}$ , we would still be left with properly characterizing the return value of  $f_{(\mathbf{a}/\mathbf{b})}(\mathbf{M})$ . Overcoming this last hurdle is even more challenging.

The preceding discussion paints a picture which shows that the generalization of the algebraic framework for dealing with default modal systems is not a straightforward exercise. In a sense, this picture portrays a limitation of the current framework. But not all is lost. The problem originates from the fact that we need to deal with classes of  $\boxminus$ -modal algebras. The situation would be much easier if we were capable of dealing with a single  $\boxminus$ -modal algebra at a time. This suggests that defaults should be internalized in the algebraic language. We conclude this section by taking some steps in this direction. In particular, we show how some of the constructions used in Def. 3.8 could be extended to a more abstract setting via suitable congruences.

**Definition 3.12.** Let  $\mathbf{L}^\Phi$  be a Lindenbaum-Tarski  $\boxminus$ -modal algebra and  $\chi$  a formula; define  $[\varphi_1]_\Phi \equiv_\chi [\varphi_2]_\Phi$  iff  $[\varphi_1]_\Phi *_\Phi [\boxminus\chi]_\Phi = [\varphi_2]_\Phi *_\Phi [\boxminus\chi]_\Phi$ .

Def. 3.12 is a step towards treating the application of default as a device for obtaining a  $\boxminus$ -modal algebra  $\mathbf{M}$  updated by the element  $[\chi]_\Phi$  in  $\mathbf{L}^\Phi$ . The updated  $\boxminus$ -modal algebra  $\mathbf{M}$  is meant to be obtained as a quotient algebra modulo the congruence  $\equiv_\chi$ . Prop. 3.6 shows that  $\equiv_\chi$  indeed is a congruence.

**Proposition 3.6.** The relation  $\equiv_\chi$  is a congruence on  $\mathbf{L}^\Phi$ .

*Proof.* That  $\equiv_\chi$  is an equivalence relation is immediate. To improve notation we drop the subscript  $\Phi$ . We need to show that: if  $[\varphi_1] \equiv_\chi [\varphi_2]$  and  $[\varphi_3] \equiv_\chi [\varphi_4]$ , then,  $[\varphi_1] * [\varphi_3] \equiv_\chi [\varphi_2] * [\varphi_4]$ ;  $-[\varphi_1] \equiv_\chi -[\varphi_2]$ ;  $f^\square([\varphi_1]) \equiv_\chi f^\square([\varphi_2])$ ; and  $f^\boxminus([\varphi_1]) \equiv_\chi f^\boxminus([\varphi_2])$ . The proof continues by cases (we only show the cases  $f^\square$  and  $f^\boxminus$ , the rest are routine):

$$\begin{array}{ll}
f^\square([\varphi_1]) * [\boxminus\chi] & f^\boxminus([\varphi_1]) * [\boxminus\chi] \\
\geq f^\square([\varphi_1] * [\boxminus\chi]) * [\boxminus\chi] & = f^\boxminus([\varphi_1]) * [\boxminus\boxminus\chi] \\
= f^\square([\varphi_2] * [\boxminus\chi]) * [\boxminus\chi] & = f^\boxminus([\varphi_1]) * f^\boxminus([\boxminus\chi]) \\
= f^\square([\varphi_2]) * (f^\square([\boxminus\chi]) * [\boxminus\chi]) & = f^\boxminus([\varphi_1] * [\boxminus\chi]) \\
\geq f^\square([\varphi_2]) * [\boxminus\chi] & = f^\boxminus([\varphi_2] * [\boxminus\chi]) \\
& = f^\boxminus([\varphi_2]) * f^\boxminus([\boxminus\chi]) \\
& = f^\boxminus([\varphi_2]) * [\boxminus\boxminus\chi] \\
& = f^\boxminus([\varphi_2]) * [\boxminus\chi].
\end{array}$$

**Proposition 3.7.** The quotient algebra  $\mathbf{L}^\Phi / \equiv_\chi$  is isomorphic to  $\mathbf{L}^{\Phi, \chi}$ .

*Proof.* Observe that  $\Phi, \chi \vdash (\varphi_1 \leftrightarrow \varphi_2)$  iff  $\Phi \vdash (\varphi_1 \wedge \boxminus\chi \leftrightarrow \varphi_2 \wedge \boxminus\chi)$ . The isomorphism between  $\mathbf{L}^\Phi / \equiv_\chi$  and  $\mathbf{L}^{\Phi, \chi}$  is given by mappings  $\iota_1$  and  $\iota_2$  defined as:  $\iota_1([\varphi]_\Phi / \equiv_\chi) = [\varphi]_{\Phi, \chi}$ ; and  $\iota_2([\varphi]_{\Phi, \chi}) = [[\varphi]_\Phi]_{\equiv_\chi}$ .

The isomorphism in Prop. 3.7 shows that the relation  $\equiv_\chi$  yields the “correct” congruence if the application of a default is to be seen as updating a  $\boxminus$ -modal algebra. Moreover, it is possible to define a function  $\varepsilon : \mathbf{L}^\Phi / \equiv_\chi \rightarrow \mathbf{L}^\Phi$  defined by  $\varepsilon([\varphi]_\Phi / \equiv_\chi) = [\varphi]_\Phi *_\Phi [\chi]_\Phi$ . The image of  $\varepsilon$  is also isomorphic to  $\mathbf{L}^{\Phi, \chi}$ . This alternative take on the effect of “applying a default” from an algebraic perspective opens a pathway on how to lift the constructions in Defs. 3.8 and 3.10 to the setting of arbitrary  $\boxminus$ -modal algebras and to connect default modal systems with logics of updates. Evidently, this is a task that needs to be fully worked out. Nonetheless, we consider that dealing with congruences is a more adequate way of exploring the meta-logical properties of default modal systems from an algebraic perspective.



### 3.5 An Application of the Algebraic Framework

The results in Sec. 3.3 shows us how to deal with default modal systems using algebraic tools. Interestingly, this enables us to formulate and obtain a completeness result using LT  $\boxminus$ -modal algebras. The algebraic machinery in Sec. 3.3 also opens a pathway to investigate other properties of default modal systems. In this section, we show how the notion of bisimulation for modal logics can be adapted and extended to enable us to compare default modal systems.

We begin by recalling the standard definition of Kripke models, and the semantics of modal formulas.

**Definition 3.13.** A *frame* is a tuple  $\mathfrak{F} = \langle W, R \rangle$  where:  $W$  is a set of elements (called *worlds*); and  $R \subseteq W^2$  is the *accessibility relation*. A *Kripke model* is a pair  $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$  where:  $\mathfrak{F}$  is a frame, and  $V : \text{Prop} \rightarrow 2^W$  is the *valuation function*. For  $w \in W$ , the pair  $\mathfrak{M}, w$  is called a *pointed model*.

**Definition 3.14.** Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a Kripke model,  $w \in W$ , and  $\varphi \in \text{Form}$ ; the *satisfiability relation*  $\mathfrak{M}, w \Vdash \varphi$  is defined according to the following rules:

$$\begin{array}{ll} \mathfrak{M}, w \Vdash p_i & \text{iff } w \in V(p_i) \\ \mathfrak{M}, w \Vdash \neg\varphi & \text{iff } \mathfrak{M}, w \not\Vdash \varphi \\ \mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff } \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \Box\varphi & \text{iff for all } w' \in W, wRw' \text{ implies } \mathfrak{M}, w' \Vdash \varphi \\ \mathfrak{M}, w \Vdash \boxminus\varphi & \text{iff for all } w' \in W, \mathfrak{M}, w' \Vdash \varphi. \end{array}$$

A Kripke model  $\mathfrak{M} = \langle W, R, V \rangle$  *satisfies* a formula  $\varphi$  at a world  $w \in W$  iff  $\mathfrak{M}, w \Vdash \varphi$ ; and it *validates*  $\varphi$ , written  $\mathfrak{M} \Vdash \varphi$ , iff  $\mathfrak{M}, w \Vdash \varphi$ , for all  $w \in W$ . The model  $\mathfrak{M}$  *satisfies* a set of formulas  $\Phi$  at  $w$ , notation  $\mathfrak{M}, w \Vdash \Phi$ , if  $\mathfrak{M}, w \Vdash \varphi$  for all  $\varphi \in \Phi$ . And it *validates*  $\Phi$ , notation  $\mathfrak{M} \Vdash \Phi$ , if  $\mathfrak{M}, w \Vdash \Phi$  for all  $w \in W$ .

The following proposition links Kripke models and the algebraic structures introduced in Sec. 2.3 (the full details can be found, e.g., in [35]).

**Definition 3.15.** Let  $\mathbf{M} = \langle B, *, -, 1, f^\Box, f^\boxminus \rangle$  be a  $\boxminus$ -modal algebra; its dual, written  $\mathbf{M}^\bullet$ , is a frame  $\langle \text{Uf}(\mathbf{M}), R \rangle$  where:  $\text{Uf}(\mathbf{M})$  is the set of all ultrafilters in  $\mathbf{M}$  and  $R$  is defined by  $u_1 R u_2$  iff  $-f^\Box(-a) \in u_1$  for all  $a \in u_2$ . The dual of an interpretation  $v : \mathbf{F} \rightarrow \mathbf{M}$  is a function  $v^\bullet : \text{Prop} \rightarrow 2^{\text{Uf}(\mathbf{M})}$  defined as:  $v^\bullet(p) = \{ u \in \text{Uf}(\mathbf{M}) \mid v(p) \in u \}$ . We define  $(\mathbf{M}, v)^\bullet = (\mathbf{M}^\bullet, v^\bullet)$ .

**Proposition 3.8.** Let  $\mathbf{M}$  be a  $\boxminus$ -modal algebra and  $v : \mathbf{F} \rightarrow \mathbf{M}_i$  be an interpretation on  $\mathbf{M}$ ; for all equations  $\varphi \approx \psi$ :

$$\mathbf{M}, v \models \varphi \approx \psi \text{ iff } (\mathbf{M}, v)^\bullet \Vdash \varphi \leftrightarrow \psi.$$

Let us recall the standard notion of bisimulation for Kripke models [7].

**Definition 3.16.** For  $i \in \{1, 2\}$ , let  $\mathfrak{M}_i = \langle W_i, R_i \subseteq W_i^2, V_i : \text{Prop} \rightarrow 2^{W_i} \rangle$  be a Kripke model; a (non-empty) relation  $Z \subseteq W_1 \times W_2$  is a *bisimulation* between  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  iff  $w_1 Z w_2$  implies

- (atom)  $w_1 \in V_1(p)$  iff  $w_2 \in V_2(p)$ , for all  $p \in \text{Prop}$ ;
- (forth) if  $w_1 R_1 w_3$ , there is  $w_4 \in W_2$  s.t.  $w_2 R_2 w_4$  and  $w_3 Z w_4$ ;
- (back) if  $w_2 R_2 w_4$ , there is  $w_3 \in W_1$  s.t.  $w_1 R_1 w_3$  and  $w_3 Z w_4$ .

The bisimulation  $Z$  is total iff: for all  $w_1 \in W_1$ , exists  $w_2 \in W_2$  s.t.  $w_1 Z w_2$ ; and for all  $w_2 \in W_2$ , exists  $w_1 \in W_1$  s.t.  $w_1 Z w_2$ . We write  $\mathfrak{M}_1 \Leftrightarrow \mathfrak{M}_2$  iff there is a total bisimulation  $Z$  between  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ ; and  $\mathfrak{M}_1, w_1 \Leftrightarrow \mathfrak{M}_2, w_2$  iff  $w_1 Z w_2$ .

Prop. 3.9 states a well-known result regarding bisimulations.

**Proposition 3.9.** If  $\mathfrak{M}_1 \Leftrightarrow \mathfrak{M}_2$ , then,  $\mathfrak{M}_1 \Vdash \varphi$  iff  $\mathfrak{M}_2 \Vdash \varphi$ .

Prop. 3.9 has an analogous proposition in terms of  $\boxminus$ -modal algebras.

**Definition 3.17.** Let  $\mathbf{M}_1$  and  $\mathbf{M}_2$  be two  $\boxminus$ -modal algebras and  $v_1$  and  $v_2$  be interpretations on  $\mathbf{M}_1$  and  $\mathbf{M}_2$ , respectively; we write  $(\mathbf{M}_1, v_1) \Leftrightarrow (\mathbf{M}_2, v_2)$  iff  $(\mathbf{M}_1, v_1)^\bullet \Leftrightarrow (\mathbf{M}_2, v_2)^\bullet$ .

**Proposition 3.10.** Let  $\mathbf{M}_1$  and  $\mathbf{M}_2$  be two  $\boxminus$ -modal algebras and  $v_1$  and  $v_2$  be interpretations on  $\mathbf{M}_1$  and  $\mathbf{M}_2$ , respectively; if  $(\mathbf{M}_1, v_1) \Leftrightarrow (\mathbf{M}_2, v_2)$ , then,  $\mathbf{M}_1, v_1 \vDash \varphi \approx \psi$  iff  $\mathbf{M}_2, v_2 \vDash \varphi \approx \psi$ .

We are now in a position to show how to extend and adapt the concept of bisimulation to the algebraic treatment of defaults and extensions.

**Definition 3.18.** Let  $\mathbf{D}_1$  and  $\mathbf{D}_2$  be the default monoids associated to two default modal systems built on the same underlying modal system; a (non-empty) relation  $Z \subseteq \mathcal{L} \times \mathcal{L}$  is a *default bisimulation* iff  $\mathbf{L}_1 Z \mathbf{L}_2$  implies:

- (atom)  $(\mathbf{L}_1, v_1) \Leftrightarrow (\mathbf{L}_2, v_2)$  where  $v_i : \mathbf{F} \rightarrow \mathbf{L}_i$  is the canonical interpretation;
- (forth) for all  $d_1 \in |\mathbf{D}_1|$ , there is  $d_2 \in |\mathbf{D}_2|$  s.t.  $d_1(\mathbf{L}_1) Z d_2(\mathbf{L}_2)$ ;
- (back) for all  $d_2 \in |\mathbf{D}_2|$ , there is  $d_1 \in |\mathbf{D}_1|$  s.t.  $d_1(\mathbf{L}_1) Z d_2(\mathbf{L}_2)$ .

We write  $\mathbf{D}_1 \Leftrightarrow \mathbf{D}_2$  iff there is a default bisimulation  $Z$  between  $\mathbf{D}_1$  and  $\mathbf{D}_2$ ; and  $\mathbf{D}_1, \mathbf{L}_1 \Leftrightarrow \mathbf{D}_2, \mathbf{L}_2$  iff  $\mathbf{L}_1 Z \mathbf{L}_2$ .

The definition of a default bisimulation in Def. 3.18 enables us to compare when two default monoids are indistinguishable; or, what is the same, it enables us to compare when two default modal systems are indistinguishable. This fact is stated in Prop. 3.11.

**Proposition 3.11.** If  $\mathbf{D}_1, \mathbf{L}_1 \Leftrightarrow \mathbf{D}_2, \mathbf{L}_2$ , then  $\mathbf{L}_1 \approx_1 \varphi \approx \psi$  iff  $\mathbf{L}_2 \approx_2 \varphi \approx \psi$ ; where  $\approx_i$  is the relation in Def. 3.11 formulated w.r.t.  $\mathbf{D}_i$ .

*Proof.* Let  $Z$  be a default bisimulation between  $\mathbf{D}_1$  and  $\mathbf{D}_2$ , and  $\mathbf{L}_1 Z \mathbf{L}_2$ . We prove the left-to-right direction, i.e., if  $\mathbf{L}_1 \approx_1 \varphi \approx \psi$ , then,  $\mathbf{L}_2 \approx_2 \varphi \approx \psi$ . Assume  $\mathbf{L}_1 \approx_1 \varphi \approx \psi$ . There is  $d_1 \in |\mathbf{D}_1|$  s.t.  $d_1(\mathbf{L}_1) \vDash \varphi \approx \psi$ . In particular,  $d_1(\mathbf{L}_1), v_1 \vDash \varphi \approx \psi$  for  $v_1 : \mathbf{F} \rightarrow f(\mathbf{L}_1)$  the canonical interpretation on  $\mathbf{L}_1$ . From (forth), there is  $d_2 \in |\mathbf{D}_2|$  s.t.  $d_1(\mathbf{L}_1) Z d_2(\mathbf{L}_2)$ . Then, from (atom), we obtain  $(d_1(\mathbf{L}_1), v_1) \Leftrightarrow (d_2(\mathbf{L}_2), v_2)$  for  $v_2 : \mathbf{F} \rightarrow f_2(\mathbf{L}_2)$  the canonical interpretation on  $\mathbf{L}_2$ . Using Prop. 3.10, it follows that  $d_2(\mathbf{L}_2), v_2 \vDash \varphi \approx \psi$ . Since canonical interpretations are closed under substitution,  $f_2(\mathbf{L}_2) \vDash \varphi \approx \psi$ . Therefore,  $\mathbf{L}_2 \approx_2 \varphi \approx \psi$ . The right-to-left direction follows the same argument using (back) instead of (forth).

We conclude this section with a comment on an application of the concept of bisimulation between default monoids to the problem of equivalence of so-called *default theories*. In the area of Default Logic, a default theory is a pair

$\langle \Phi, \Delta \rangle$  where  $\Phi$  is a set of formulas and  $\Delta$  is a set of defaults. The problem of equivalence of default theories pertains to the following question: in which sense two default theories  $\langle \Phi_1, \Delta_1 \rangle$  and  $\langle \Phi_2, \Delta_2 \rangle$  can be regarded as equivalent? Adequate answers to this question have a bearing in the setting of transformation of logical programs, e.g., to improve their efficiency – it is well-known that default theories may be seen as the logical counterpart of logical programs. It is common to address equivalence of default theories from a syntactic point of view by focusing on what is the case w.r.t. extensions (see, e.g., [30, 21]). Under such a point of view, a default theory  $\langle \Phi_1, \Delta_1 \rangle$  is deemed equivalent to a default theory  $\langle \Phi_2, \Delta_2 \rangle$  whenever  $\mathbf{E}_{\Delta_1}^{\Phi_1} = \mathbf{E}_{\Delta_2}^{\Phi_2}$ ; and they are deemed to be strongly equivalent whenever  $\langle \Phi_1 \cup \Phi_3, \Delta_1 \cup \Delta_3 \rangle$  and  $\langle \Phi_2 \cup \Phi_3, \Delta_2 \cup \Delta_3 \rangle$  are equivalent for all default theories  $\langle \Phi_3, \Delta_3 \rangle$ . By way of example,  $\langle \{\Box p\}, \emptyset \rangle$  and  $\langle \emptyset, \{\top/\Box p\} \rangle$  are equivalent, but they are not strongly equivalent (as  $\langle \{\Diamond \neg p\}, \emptyset \rangle$  distinguishes them). The concept of bisimulation between default monoids enables us to look at the problem of equivalence of default theories in a new light, i.e., from a semantic perspective. For a fixed underlying modal system, we could say that  $\langle \Phi_1, \Delta_1 \rangle$  and  $\langle \Phi_2, \Delta_2 \rangle$  are equivalent whenever  $\mathbf{D}_1, \mathbf{L}^{\Phi_1} \Leftrightarrow \mathbf{D}_2, \mathbf{L}^{\Phi_2}$  (where  $\mathbf{D}_1$  and  $\mathbf{D}_2$  are the default monoids associated to the default modal systems constructed relative to  $\Delta_1$  and  $\Delta_2$ , respectively). The notion of strong equivalence is defined in a similar way. To be noted, our definition of equivalence is given in semantic terms building on the notion of bisimulation for the underlying modal system. To be noted also, our definition of equivalence pays closer attention to the different elements of a default theory (i.e., its set of formulas and its set of defaults). For instance, notice that under our definition,  $\langle \{\Box p\}, \emptyset \rangle$  and  $\langle \emptyset, \{\top/\Box p\} \rangle$  are not equivalent, as  $\mathbf{L}^{\{\Box p\}}, v_1 \not\equiv \mathbf{L}^{\emptyset}, v_2$  for  $v_1$  and  $v_2$  the canonical interpretations on the corresponding LT  $\boxminus$ -modal algebras. Looking at problems of equivalence from a semantic perspective has a long standing tradition in the field of Modal Logic. In this respect, bisimulations have proven to be a useful tool. The concept of bisimulation between default monoids is a step in this direction.

It is worth noting that a practical advantage of bisimulations over other techniques for proving semantic equivalence is given by checking properties relative to given points. In our case, this boils down to prove properties of the Lindenbaum-Tarski algebras. This has a “local” flavor that is in contrast to the usual methods for proving equivalence between default theories. They are, in some sense, “global” in nature, as they need to look into the entire collection of extensions. Furthermore, it is worth noting that bisimulations can be algebraized via coalgebras. In this respect, our work seems a natural step towards the definition of an algebraic toolset for reasoning about default modal systems in a mathematical setting.

As further work also, it remains to generalize the concept of bisimulation between default monoids along two main lines: default operators defined relative to arbitrary defaults, and arbitrary  $\boxminus$ -modal algebras. This generalization seems an ideal tool for discussing the deductive aspects of default theories independently of their particular syntactical constructs.

## 4 On Defaults as Model Updates

In this section, we begin to explore a connection between our algebraic approach for default modal systems and (single agent) Public Announcement Logic (PAL). This connection is inspired by the results presented in [24, 25].

To set up context for discussion, we briefly introduce some basic notions of PAL (see, e.g., [26] for details). As a modal logic, PAL extends the modal logic S5 (seen as the logic of knowledge) with a new modality  $[!\psi]\varphi$  of announcement. Given a Kripke model  $\mathfrak{M}$ , with  $w$  one of its worlds, this modality is defined as:

$$\mathfrak{M}, w \Vdash [!\psi]\varphi \quad \text{iff} \quad \mathfrak{M}, w \Vdash \psi \text{ implies } \mathfrak{M}_{|\psi}, w \Vdash \varphi, \quad (3)$$

where  $\mathfrak{M}_{|\psi}$  is the restriction of  $\mathfrak{M}$  to those worlds in which  $\psi$  holds. Intuitively, a formula  $[!\psi]\varphi$  states that  $\varphi$  holds whenever  $\psi$  is announced truthfully. Model theoretically, the interpretation of announcing  $\psi$  relativizes the model in which  $\psi$  is announced to the submodel in which  $\psi$  holds everywhere. The formula  $\varphi$  is then evaluated on the relativized model. To better explain our ideas, without loss of generality, we choose to look at the “knowledge” modality of PAL as the “universal” modality  $\boxplus$ . Moreover, we consider the “existential” modality  $\boxtimes$  defined as an abbreviation of  $\neg\boxplus\neg\varphi$  and understood as:  $\varphi$  holds somewhere in the model.

There are some interesting similarities between announcements in PAL and defaults. For instance, in [24, 25], an announcement is presented as a homomorphism between the modal algebra in which the announcement occurs, and the modal algebra corresponding to the submodel in which the announced formula holds. The algebraic machinery introduced in Sec. 3.3 for defaults sets the basis for thinking about the application of defaults as a logic of updates. More precisely, we may construe the algebraic semantics of a default  $\delta = \pi/\chi$  as an update from the LT  $\boxplus$ -modal algebra in which the default is considered, say  $\mathbf{L}^\Phi$ , to the LT  $\boxplus$ -modal algebra updated with the consequent of the default,  $\mathbf{L}^{\Phi,\chi}$ , if the default is applicable. It is possible to prove that there are homomorphisms from  $\mathbf{L}^{\Phi,\chi}$  to  $\mathbf{L}^\Phi$  and from  $\mathbf{L}^\Phi$  to  $\mathbf{L}^{\Phi,\chi}$ . This allows us to look at  $\mathbf{L}^{\Phi,\chi}$  as the algebraic counterpart of the update caused by  $\delta$ . Clearly, a “default” update takes place only if the prerequisite of the default is provable and its consequent does not yield an inconsistency. But this situation is not far apart from the case of announcements; where the update takes place only if the formula being announced is true (see Eq. (3)). In both cases, that of an announcement and that of the algebraic application of a default, the update is captured by a homomorphism from the original modal algebra to an updated modal algebra obtained as a quotient construction. The quotient construction for the default case is explored by the end of Sec. 3.4.

The similarities between announcements in PAL and defaults are even more apparent when contrasted with the way in which announcements are dealt with in [24]. Getting into some level of detail, the approach in [24] exploits the duality between models and modal algebras. A formula  $\psi$  is interpreted as an element  $b$  in an S5 modal algebra  $\mathbf{M} = \langle B, *, -, f^\square \rangle$ . The result of announcing this formula is a modal algebra constructed as a quotient modulo a congruence  $\equiv_b$  defined as:

$$b_1 \equiv_b b_2 \quad \text{iff} \quad b_1 * b = b_2 * b.$$

This congruence bears a close resemblance to the one we explore by the end of

Sec. 3.4. The main difference between this congruence and ours rests on the fact that the former is presented in the setting of **S5**, whereas ours is presented in a setting where global modal consequence is taken as the basis on which to build default modal systems. This said, the approach in [24] is more abstract than ours; since it considers arbitrary modal algebras and not just those obtained via Lindenbaum-Tarski constructions.

It turns out that, looking in detail what occurs with defaults and announcements in a very simple setting spearheads an interesting and promising research direction. More precisely, suppose that we admit and work on a default modal system built on Classical Propositional Logic (CPL). Further, suppose that the (credulous) default consequence relation is defined as a relation between pairs of formulas; i.e.,  $\varphi \vdash \psi$  indicates that  $\psi$  is a credulous default consequence of  $\varphi$ . Even further, suppose that there is only a single default  $\delta = \pi/\chi$ . It is possible to establish the following result:

$$\varphi \vdash \psi \quad \text{iff} \quad \mathfrak{C} \Vdash [!\varphi]\psi \vee [!\varphi](\pi \wedge \Phi\chi \wedge [!\chi]\psi),$$

where  $\mathfrak{C} = \{w \mid w \subseteq \mathbf{Prop}\}$  (i.e.,  $\mathfrak{C}$  is a Kripke model whose worlds are in one-to-one correspondence with all possible propositional valuations). To see why the results holds, notice that  $\varphi \vdash \psi$  iff  $\varphi \rightarrow \psi$  is a tautology in CPL, or  $\varphi \wedge \chi \rightarrow \psi$  is a tautology in CPL; i.e., either we can find a proof of  $\psi$  from  $\varphi$ , or we can find a proof of  $\psi$  from  $\varphi$  using  $\chi$  as an additional assumption. The first case corresponds to not using the default in the proof. The second case corresponds to using the default in the proof. On the other hand,  $\mathfrak{C} \Vdash [!\varphi]\psi \vee [!\varphi](\pi \wedge \Phi\chi \wedge [!\chi]\psi)$  iff for all  $w \in \mathfrak{C}$ , we have  $\mathfrak{C}, w \Vdash [!\varphi]\psi$  or  $\mathfrak{C}, w \Vdash [!\varphi](\pi \wedge \Phi\chi \wedge [!\chi]\psi)$ . The case  $\mathfrak{C}, w \Vdash [!\varphi]\psi$  corresponds to  $\varphi \rightarrow \psi$  being a tautology in CPL. The case  $\mathfrak{C}, w \Vdash [!\varphi](\pi \wedge \Phi\chi \wedge [!\chi]\psi)$  corresponds to being able to use the default and  $\varphi \wedge \chi \rightarrow \psi$  being a tautology in CPL. The interesting part here is how the application of a default is captured in the object language of PAL. The provability of  $\pi$  is captured by checking what is the case in every possible  $\varphi$ -world of  $\mathfrak{C}$  (which is, in fact, an assignment in CPL in which  $\varphi$  holds); and the non-provability of  $\chi$  is captured by checking that there is in fact a  $\varphi$ -world in which  $\chi$  holds (which is tantamount to checking consistency in CPL relative to  $\varphi$ ). Finally, what is the case after applying an applicable default is captured as an announcement. Notice that in this case working in  $\mathfrak{C}$  is necessary (the result would not hold if we take a strict subset of  $\mathfrak{C}$ ). It is worth noting also that in this case default consequence becomes a model checking problem. These are some preliminary ideas that need to be worked out in detail. In particular, it is necessary to look at how to deal not only with a single formula  $\varphi$  but with sets of formulas  $\Phi$ ; and how to deal not only with a single default  $\delta$  but with a set of defaults  $\Delta$ . The latter also involve dealing with possible combinations in the application of defaults; potentially involving blow up in the size of the formula. Also important is to learn how to apply these ideas beyond CPL for modal systems. It can easily be seen that is not an straightforward exercise – for a start, announcements are not closed under certain classes of models (e.g., in those that are *serial*).

The discussion above offers only some first steps in understanding the relationship between defaults and updates: both in terms of a full algebraization of default modal systems, and in terms of establishing a tight connection with logics of updates. In working towards a full algebraization of default modal systems, we would like to interpret the application of a default over arbitrary modal algebras, and not only as an update over LT  $\boxminus$ -modal algebras. Internalizing

the application of the default and its effect in the algebraic language (perhaps as a new operator) seems to be a path to explore. Along this line, it is necessary also to work out how to capture the application of a sequence of defaults needed to build an extension. Moreover, it would also be interesting to know whether it is possible to develop a class of algebraic structures for default modal systems parallel to the class of modal algebras for modal systems. In turn, in what refers to establishing a tight connection with logics of updates, it would be interesting to be able to prove a reduction result between a default modal system and a logic of announcements (or establishing a difference in expressive power between one and the other). The challenge is deciding on an adequate logic of announcements and in finding whether it is possible to faithfully translate the application of a default as a form of update in this logic. Finally, upon defining the semantics of defaults as updates, we would like to study defaults as dynamic epistemic operators. In particular, we would like to explore whether defaults can be used to represent some novel form of communication between agents in a multi-agent setting.

## 5 Final Remarks

We presented a family of default logics built on modal logics ranging from  $\mathbf{K}$  to  $\mathbf{S5}$ , and studied some of their properties. We approached this presentation syntactically via what we called a default modal system. For each default modal system we formulated a notion of default deducibility to make explicit how defaults interact with deducibility in the underlying modal system. Then, we offered an algebraic treatment of defaults and extensions, via transformations on  $\text{LT } \boxplus$ -modal algebras and default monoids, respectively. This algebraic treatment enabled us to obtain an algebraic completeness result. Interestingly enough, this approach also enabled us to think of a way of comparing default logics by borrowing ideas from the concept of a bisimulation in modal logic. To our knowledge, this is the first work addressing default logic algebraically.

There are several interesting lines for future work. We do notice that our work is not an algebraization of a logic. Instead, we have taken advantage of algebraic tools to study default modal systems from a semantic perspective. In this respect, our work is a first step towards an algebraization of default modal systems. There is still a need to identify a class of algebras for default modal systems that would play a role akin to  $\boxplus$ -modal algebras in modal systems. We also wish to generalize our constructions and results to arbitrary  $\boxplus$ -modal algebras and not just  $\text{LT } \boxplus$ -modal algebras. We consider that the trail of coalgebras (see [36]) may provide an adequate abstract framework in which to generalize and further investigate our ideas. Bisimulations can be algebraized via coalgebras (see e.g., [1, 29]). In this respect, our work on bisimulations between default monoids is a natural step towards the definition an algebraic toolset for reasoning about default modal systems in a mathematical setting. More precisely, coalgebras provide an abstract way of formalizing bisimulation in the realm of Category Theory. Roughly speaking, cospans provide an abstract version of relations. In such a setting, bisimulations can coalgebraically be captured as arrows  $R \rightarrow \mathbf{F}(R)$ , where  $\mathbf{F}$  is an appropriate endofunctor providing the structure needed for capturing behavior. In our case, we have used monoids to capture the closure under the application of defaults (with the aim of “building” exten-

sions). We would like to see these monoids as relations and their bisimulations as coalgebras. One of the main difficulties in using the ideas of coalgebras in our framework is how to capture the closure under the application of defaults as an endofunctor. This line of research is part of our further work efforts. Furthermore, we wish to exploit our algebraic treatment of default modal systems to study semantic properties such as: invariance and Hennessy-Milner theorems, interpolation, Beth definability, etc.

Lastly, our characterization of defaults as transformations on  $\boxplus$ -modal algebras works as a sort of “update”. It would be interesting to find connections with algebraic approaches to logics with update operators, in the sense of e.g. Public Announcement Logic. The work reported in [24, 25] seems to shed some light in this direction.

**Acknowledgments.** Our work is supported by ANPCyT-PICT-2020-3780, CONICET project PIP 11220200100812CO, and by the LIA SINFIN.

## References

- [1] N. Aczel, P. annd Mendler. A final coalgebra theorem. In D. H. Pitt, D. E. Rydeheard, P. Dybjer, A-M. Pitts, and A. Poigné, editors, *Category Theory and Computer Science*, pages 357–365, Berlin, Heidelberg, 1989. Springer Berlin Heidelberg.
- [2] T. Ågotnes and N. Alechina. The dynamics of syntactic knowledge. *Journal of Logic and Computation*, 17(1):83–116, 2007.
- [3] G. Antoniou. *Nonmonotonic Reasoning*. The MIT Press, Cambridge, 1997.
- [4] G. Antoniou and K. Wang. Default logic. In D. Gabbay and J. Woods, editors, *The Many Valued and Nonmonotonic Turn in Logic*, volume 8 of *Handbook of the History of Logic*, pages 517–555. North-Holland, 2007.
- [5] F. Baader and B. Hollunder. Embedding defaults into terminological knowledge representation formalisms. In *Proceedings of KR ’92*, pages 306–317. Morgan Kaufmann, 1992.
- [6] P. Blackburn, M. de Rijke, and Y. Venema, editors. *Modal Logic*. Cambridge University Press, 2002.
- [7] P. Blackburn and J. van Benthem. Modal logic: A semantic perspective. In Blackburn et al. [8], pages 1–84.
- [8] P. Blackburn, J. van Benthem, and F. Wolter, editors. *Handbook of Modal Logic*. Elsevier, 2007.
- [9] V. Cassano, R. Fervari, C. Areces, and P. F. Castro. Interpolation and beth definability in default logics. In *Proceedings of JELIA 2019*, volume 11468 of *LNCS*, pages 675–691. Springer, 2019.
- [10] V. Cassano, R. Fervari, C. Areces, and P. F. Castro. Default modal systems as algebraic updates. In *Proceedings of DaLi 2020*, volume 12569 of *LNCS*, pages 103–119. Springer, 2020.

- [11] J. Delgrande, T. Schaub, and W. Jackson. Alternative approaches to default logic. *Artificial Intelligence*, 70(1-2):167–237, 1994.
- [12] H. N. Duc. Reasoning about rational, but not logically omniscient, agents. *Journal of Logic and Computation*, 7(5):633–648, 1997.
- [13] R. Fervari and F. R. Velázquez-Quesada. Dynamic epistemic logics of introspection. In *Proceedings of DALI 2017*, volume 10669 of *LNCS*, pages 82–97. Springer, 2017.
- [14] R. Fervari and F. R. Velázquez-Quesada. Introspection as an action in relational models. *Journal of Logical and Algebraic Methods in Programming*, 108:1–23, 2019.
- [15] M. Fitting. Modal proof theory. In Blackburn et al. [8], pages 85–138.
- [16] C. Froidevaux and J. Mengin. Default logics: A unified view. *Computational Intelligence*, 10:331–369, 1994.
- [17] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyashev. *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.
- [18] S. Givant and P. Halmos. *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. Springer, 2009.
- [19] V. Goranko and S. Passy. Using the universal modality: Gains and questions. *Journal of Logic and Computation*, 2(1):5–30, 1992.
- [20] D. Grossi and F. R. Velázquez-Quesada. Syntactic awareness in logical dynamics. *Synthese*, 192(12):4071–4105, 2015.
- [21] K. Inoue and C. Sakama. Generality and equivalence relations in default logic. In *Proceedings of AAI 2007*, pages 434–439. AAAI Press, 2007.
- [22] M. Jago. Epistemic logic for rule-based agents. *Journal of Logic, Language and Information*, 18(1):131–158, 2009.
- [23] W. Łukaszewicz. Considerations on default logic: An alternative approach. *Computational Intelligence*, 4:1–16, 1988.
- [24] M. Ma. Mathematics of public announcements. In *Proceedings of LORI 2011*, pages 193–205. Springer Berlin Heidelberg, 2011.
- [25] M. Ma, A. Palmigiano, and M. Sadrzadeh. Algebraic semantics and model completeness for intuitionistic public announcement logic. *Annals of Pure and Applied Logic*, 165(4):963–995, 2014.
- [26] J. Plaza. Logics of public communications. *Synthese*, 158(2):165–179, 2007.
- [27] R. Reiter. A logic for default reasoning. *Artificial Intelligence*, 13(1-2):81–132, 1980.
- [28] A. Solaki. Steps out of logical omniscience. Master’s thesis, Institute for Logic, Language and Computation, Amsterdam, 2017. MoL-2017-12.



- [29] S. Staton. Relating coalgebraic notions of bisimulation. *Logical Methods in Computer Science*, 7(1), 2011.
- [30] H. Turner. Strong equivalence for logic programs and default theories (made easy). In *Proceedings of LPNMR 2001*, volume 2173 of *LNCS*, pages 81–92. Springer, 2001.
- [31] J. van Benthem and E. Pacuit. Dynamic logics of evidence-based beliefs. *Studia Logica*, 99(1):61–92, 2011.
- [32] J. van Benthem and F. R. Velázquez-Quesada. The dynamics of awareness. *Synthese (Knowledge, Rationality and Action)*, 177(Supplement 1):5–27, 2010.
- [33] M. Y. Vardi. On epistemic logic and logical omniscience. In *Proceedings of TARK 1986*, pages 293–305. Morgan Kaufmann, 1986.
- [34] F. R. Velázquez-Quesada. Explicit and implicit knowledge in neighbourhood models. In *Proceedings of LORI 2013 Workshop*, pages 239–252, 2013.
- [35] Y. Venema. Algebras and general frames. In Blackburn et al. [6], pages 263–333.
- [36] Y. Venema. Algebras and coalgebras. In Blackburn et al. [8], pages 331–426.