Uncertainty-Based Knowing How Logic

Carlos Areces[1,2] Raul Fervari[1,2,3]
Andrés R. Saravia[1,2] Fernando R. Velázquez-Quesada[4]

[1]FAMAF, Universidad Nacional de Córdoba, Argentina
[2]Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina
[3]Guangdong Technion - Israel Institute of Technology (GTIIT), China
[4]Universitetet i Bergen, Norway

Abstract

We introduce a novel semantics for a multi-agent epistemic operator of *knowing how*, based on an indistinguishability relation between plans. Our proposal is, arguably, closer to the standard presentation of *knowing that* modalities in classical epistemic logic. We study the relationship between this new semantics and previous approaches, showing that our setting is general enough to capture them. We also study the logical properties of the new semantics. First, we define a sound and complete axiomatization. Second, we define a suitable notion of bisimulation and prove correspondence theorems. Finally, we investigate the computational complexity of the model checking and satisfiability problems for the new logic.

1 Introduction

Epistemic logic (EL; [29, 14]) is a logical formalism tailored for reasoning about the knowledge of abstract autonomous entities commonly called agents (e.g., a human being, a robot, a vehicle). It has contributed to the formal study of complex multi-agent epistemic notions not only in philosophy [26] but also in computer science [14, 42] and economics [48].

Standard epistemic logics deal with an agent's knowledge about the truth-value of propositions (the notion of *knowing that*). Thus, they focus on the study of sentences like *"the agent knows that it is sunny in Paris"* or *"the robot knows that it is standing next to a wall"*. For doing so, at the semantic level, EL formulas are typically interpreted over relational models [10, 11]: essentially, labeled directed graphs. The elements of the domain (called *states* or *worlds*) represent different possible situations, and they fix the facts that an agent might or might not know. Then, the knowledge of each agent is given by her *epistemic indistinguishability* relation, used to represent her uncertainty about the truth: related states are considered indistinguishable for the agent. Finally, an agent is said to know that a proposition φ is true at a given state w if and only if φ holds in all states she cannot distinguish from w (i.e., in all states accessible from w). In order to capture properly the properties of knowledge, it is typically assumed that the indistinguishability relation is an equivalence relation.

In spite of its simplicity, this indistinguishability-based representation of knowledge has several advantages. First, it captures the agent's *high-order* knowledge (knowledge about her own knowledge and that of other agents). Moreover, due to its generality, it opens the way to study other epistemic notions, such as the notion of *belief* [29]. Finally, it allows a very natural representation of actions through which knowledge changes [54, 51].

In recent years, other forms of knowledge have been studied (see the discussion in [59]). Some authors have studied knowledge of propositions using rather the notion of *knowing whether* [24, 15]; some others have focused on the reasons/justifications for propositional knowledge, exploring the notion of *knowing why* [6, 61]; some more have looked at more general scenarios, proposing logics for *knowing the value* [21, 7, 55]. A further and particularly interesting form of knowledge, motivated by different scenarios in philosophy and AI, is one that focuses rather on the agent's abilities: the notion of *knowing how* [16]. Intuitively, an agent knows how to achieve φ given ψ if she has the *ability* to guarantee that φ will be the case whenever she is in a situation in which ψ holds. Arguably, this notion is particularly important as it provides the formal foundations of automated planning and strategic reasoning within AI.

Historically, the concept of knowing how has been considered different from knowing that, as posed e.g. in [49]. Knowing how is often seen as a reflection of actions or abilities that agents may take, in an intelligent manner, in order to achieve a certain goal. In turn, there is a large literature connecting *knowing how* with logics of knowledge and action (see, e.g., [41, 43, 32, 53, 28]). However, the way in which these proposals represent *knowing how* has been the target of criticisms. The main issue is that a simple combination of standard operators expressing *knowing that* and *ability* (see, e.g., [52]) does not seem to lead to a natural notion of *knowing how* (see [30, 27] for a discussion).

Taking these considerations into account, [58, 59, 60] introduced a novel framework based on a binary *knowing how* modality that is not defined in terms of *knowing that*. At the semantic level, this language is also interpreted over relational models — called in this context labeled transition systems (LTSs). Yet, relations do not represent indistinguishability anymore; they rather describe the actions the agent has at her disposal (similar to what is done in, e.g., propositional dynamic logic [23]). Indeed, an edge labeled a from state w to state u indicates now that the agent can execute action a to transform state w into u. In the proposed semantics, the new modality $\mathsf{Kh}(\psi, \varphi)$ holds if and only if there is a "plan" — a sequence of actions satisfying a constraint called strong executability (SE) — leading from ψ-states to φ-states. Intuitively, SE implies that the plan is "fail-proof" in the LTS; it unerringly leads from every ψ-state only to φ-states. Other variants of this *knowing how* operator follow a similar approach (see [33, 36, 17, 57]). Further motivation for these semantics can be found in the referred papers.

It is interesting to notice how LTSs have no epistemic component: their relations are interpreted as actions, and then the abilities of an agent are defined only in terms of what these actions can achieve. This is in sharp contrast with standard EL, where relational models provide two kinds of information: ontic facts about the given situation (the model's evaluation point) and the particular way an agent 'sees' this situation (both the possible states available in the model and the agent's indistinguishability relation among them). In particular, in a multi-agent scenario, all agents share the same ontic information, and differ on their *epistemic interpretation* of it. If one wants to mirror the situation in EL, it seems natural that *knowing how* should be defined in terms of some kind of indistinguishability over the actual situation. Such an extended model would then be able to capture both the abilities of an agent as given by her available actions (the ontic information) as well as the knowledge (or lack of it) that arises from her uncertainty (the epistemic information).

This paper investigates a new semantics for $\mathsf{Kh}_i(\psi, \varphi)$, a multi-agent version of the *knowing how* modality, first presented in [4]. This semantics introduces two ideas. The first, and crucial, is the use of a notion of *epistemic indistinguishability over plans*, in the spirit of the *strategy indistinguishability* of, e.g., [31, 9]. The intuition behind it is that, under the original LTS semantics, the only reason why an agent might not know how to achieve a goal is because there are no adequate actions available. However, one can think of scenarios in which the lack of knowledge arises for a different reason: the agent might have an adequate plan available (she has the ability to do something), and yet she might not be able to distinguish it from a non-adequate one, in the sense of not being able to tell that, in general, these plans produce different outcomes. Section 4 provides a deeper discussion on this. In this way, these *uncertainty-based* LTS*s* reintroduce the notion of epistemic indistinguishability.

Now, although indistinguishability over plans is the main idea behind the new semantics, this proposal incorporates a second insight. One can also think of scenarios in which some of the actions, despite being *ontically* available, are not *epistemically* accessible to the agent. There might be several reasons for this, but an appealing one is that the agent might not be *aware of* all available actions. In such cases, the epistemically inaccessible actions are then not even under consideration when the agent looks for a plan to reach a goal. The idea of awareness is not new in the EL literature: it has been used for dealing with the problem of logical omniscience [56, 50, 22] by allowing the agent not to be aware of all involved atoms/formulas, thus bringing it closer to what a 'real' resource-bounded agent is [13].

Notice that, the ideas discussed above are in line with a reading that has a consensus among the literature (see, e.g., [25]): knowing how of an agent entails her ability (i.e., the capacity of actually doing it), but ability does not necessarily entail knowing how. It is equally important to notice that, in the new semantics, the agent does not *need* to be incapable of distinguishing certain actions, and she does not *need* to be unaware of some of them. As it will be proved, the new semantics is a generalization of the original ones in [58, 60]. Thus, in the new semantics, an agent who does not have uncertainty among plans and has full awareness of all of them is, knowledge-wise, exactly as an agent in the original semantics. Moreover: as investigated in [5], this new semantics allows the definition of operators that, by modifying the epistemic component in the models, represent changes on the agent's knowledge how.

Contributions. Our work aims to shed new light on knowing how logics. In particular, we investigate a new multi-agent semantics for capturing the notion of knowing how, generalizing previous proposals [58, 59, 60, 4]. Herein we establish, at the level of models, a distinction between ontic information shared by the agents (or abilities) and epistemic information for each individual agent (or awareness). In our semantics, knowing how is given by the latter, instead of by the former, as in existing approaches [58, 59, 60]. Moreover, we present a thorough study of the metalogical properties of the new logic, and compare it with previous approaches. Our contributions can be summarized as follows:

(1) We introduce a new semantics for $\mathsf{Kh}_i(\psi, \varphi)$ (for i an agent) that reintroduces the notion of epistemic indistinguishability from classical EL. This dimension captures the awareness for each particular agent over the available abilities in the real world.

(2) We introduce a suitable notion of bisimulation for the new semantics, based on ideas from [18, 19]. We prove an invariance result, and a Hennessy-Milner style theorem over finite models.

(3) We show that the logic obtained is strictly weaker (and this is an advantage, as we will discuss) than the logic from [58, 59, 60]. Still, the new semantics is general enough to capture the original proposal by imposing adequate conditions on the class of models. Apart from the direct correspondence between models of each framework established already in [4], we introduce a new general class of models that also does the job.

(4) We present a sound and complete axiomatization for the logic over the class of all models.

(5) We study the computational properties of our logic. First, we provide a finite model property via filtrations. I.e., we show how, given an arbitrary model, it is possible to obtain a finite model satisfying the same set of formulas. A more careful selection argument can be used to prove that the satisfiability problem for the new logic is NP-complete, whereas model checking is in P.

This paper extends [4], providing provide detailed discussions and motivations as well as full proofs. Moreover: the results about bisimulations, expressive power and finite models via filtrations are novel.

Outline of the article. Section 2 recalls briefly the literature on *knowing how*. Section 3 recalls the framework of [58, 59, 60], including its axiom system. Section 4 introduces *uncertainty-based LTSs*, indicating how they can be used for interpreting a multi-agent version of the *knowing how* language. In Section 5 we introduce a suitable notion of bisimulation, together with correspondence theorems. We provide a sound and complete axiom system in Section 6. Section 7 studies the correspondence between our semantics and the one in the original proposals. In particular, we present two different classes of models that capture the original semantics. In Section 8 we investigate a finite model property via filtrations (Subsection 8.1), and the computational complexity of model checking and the satisfiability problem for our logic (Subsection 8.2). We finish in Section 9 with some conclusions and future lines of research.

2 A short review of the literature

The ideas discussed in the previous section concerning the notion of *knowing how* introduced in [58, 59, 60] have been successful, and have lead to different works in the literature. An earlier one is [36], which considers a ternary modality $\mathsf{Kh}(\psi, \chi, \varphi)$ asking for a plan whose intermediate states satisfy χ. Then, [33] introduces a weaker binary modality $\mathsf{Kh}^{\mathsf{w}}(\psi, \varphi)$ that allows plans that abort, and the states reached by these aborted executions should also satisfy the goal φ. Finally, [57] uses a semantics under which intermediate actions in a given plan may be skipped.

The respective works introducing these variants also provide an axiom system (interestingly, the logic for the modality with skippable plans is the same as the logic for the original modality). Regarding computational behaviour, the satisfiability problem has been proved to be decidable (in their respective papers) for the basic system, the one allowing aborted executions and the one with skippable plans. For the original logic, a Σ_2^P complexity bound has been established in [3]. Finally, suitable notions of bisimulation for these systems can be found in [18, 19] (for all but the one with skippable plans) and [57] (for the one with skippable plans). These bisimilarity tools have been useful to investigate the systems' relative expressive power. It has been shown that the original binary modality $\mathsf{Kh}(\psi, \varphi)$ is strictly less expressive than the one with intermediate steps, $\mathsf{Kh}(\psi, \chi, \varphi)$, and that they are both incomparable with the modality with aborted executions, $\mathsf{Kh}^{\mathsf{w}}(\psi, \varphi)$. Moreover, in [12] the computational complexity of the model-checking problem for different knowing how logics is characterized. In particular, it is established that model-checking for the basic knowing how logic from [58, 59, 60] is PSpace-complete, whereas for a variant with budget constraints is ExpSpace-hard. Other constraints over plans are also studied therein, concretely the variant of [4] (the one studied in this paper) with regularity constraints and budgets, for which model-checking is in P. More recently, in [2], the framework of knowing how is extended to a deontic setting, formalizing the notion of *knowingly complying*.

Further proposals explore new features. For instance, a natural extension is considering the interaction between *knowing how* and standard *knowing that* modalities. In [17], a single-agent logic with the two modalities is introduced. The knowing how operator is, unlike previous approaches, a unary local modality $\mathsf{Kh}(\varphi)$, and its interpretation allows branching plans. The interaction between both kinds of knowledge is studied via an axiom system, and it is proved that its satisfiability problem is decidable. The decidability result has been recently refined in [34, 35], where PSpace-completeness is proved for the satisfiability problem, via a tableau-based procedure. In [39] a neighbourhood semantics is provided for the *knowing how* modality, as an alternative to the standard relational semantics.

Other papers incorporate multi-agent behaviour for *knowing how* and *knowing that* modalities. For instance, in [44, 46] this is explored in the context of coalitions, i.e., the logic is used to describe different notions of collective knowledge. It is known that a fragment of this logic is incomparable in expressive power with the logic from [17] (the proof uses bisimulation, and it is presented in [19]). Other variants of this logic have been explored, including those relying on *second-order knowing how* strategies [45], and *knowing how* with degrees of uncertainty [47]. Axiom systems are presented for each logic.

Finally, a multi-agent knowing how logic describing the behaviour of epistemic planning is investigated in [37]. The main peculiarity is that the execution of an action is represented by an update in the model via epistemic action models [8]. The logic obtained is strictly weaker than the one in [17].

Again, its satisfiability problem is decidable. This work is extended in [40], which provides a unified approach for planning-based knowing how. More remarkably, the work in [38] establishes a connection between planning and knowing how, not just from the perspective of *planning-based* know how, but also the other way around: a planning problem based on know how goals. To do so, the authors introduce a model checking algorithm running in P time.

3 A logic of knowing how

This section recalls the basics of the *knowing how* framework from [58, 59, 60].

Syntax and semantics. Throughout the text, let Prop be a countable non-empty set of propositional symbols.

Definition 3.1 Formulas of the language L_{Kh} are given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}(\varphi, \varphi),$$

with $p \in \mathsf{Prop}$. Boolean constants and other Boolean connectives are defined as usual. Formulas of the form $\mathsf{Kh}(\psi, \varphi)$ are read as *"when ψ holds, the agent knows how to make φ true"*. ⊣

In [58, 59, 60] (and variations like [36, 33]), formulas of L_{Kh} are interpreted over *labeled transition systems*: relational models in which the relations describe the state-transitions available to the agent.

Definition 3.2 (Actions and plans) Let Act be an enumerable set of (basic) action names, and let Act^* be the set of finite sequences over Act. Elements in Act^* are called *plans*, with ϵ being the *empty plan*. Given $\sigma \in \mathsf{Act}^*$, let $|\sigma|$ be the length of σ (note: $|\epsilon| := 0$). For a plan σ and $0 \leq k \leq |\sigma|$, the *plan* σ_k is σ's initial segment up to (and including) the kth position (with $\sigma_0 := \epsilon$). For $0 < k \leq |\sigma|$, the *action* $\sigma[k]$ is the one in σ's kth position. ⊣

Definition 3.3 (Labeled transition systems) A *labeled transition system* (LTS) over Prop is a tuple $\mathcal{S} = \langle \mathrm{W}, \mathrm{R}, \mathrm{V}, \mathsf{Act}\rangle$ where W is a non-empty set of states (also denoted by $\mathrm{D}_{\mathcal{S}}$), $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W}\times\mathrm{W} \mid a \in A,\ \text{for some } A \subseteq \mathsf{Act}\}$ is a collection of binary relations on W[1], $\mathrm{V} : \mathrm{W} \to 2^{\mathsf{Prop}}$ is a labelling function, and Act is an enumerable set of action names. Given an LTS $\mathcal{S}$ and $w \in \mathrm{D}_{\mathcal{S}}$, the pair $(\mathcal{S}, w)$ is a *pointed* LTS (parentheses are usually dropped). ⊣

An LTS describes the *abilities* of the agent; thus, sometimes (e.g., [58, 59, 60]) it is also called an *ability map*. Here we introduce some useful definitions. It is worth noticing that, although the set Act is a potentially infinite set, the relations in the model might be defined only for a (possibly finite) subset of actions.

Definition 3.4 Let $\{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in A,\ \text{for some } A \subseteq \mathsf{Act}\}$ be a collection of binary relations. Define $\mathrm{R}_\epsilon := \{(w, w) \mid w \in \mathrm{W}\}$ and, for $\sigma \in \mathsf{Act}^*$ and $a \in \mathsf{Act}$, $\mathrm{R}_{\sigma a} := \{(w, u) \in \mathrm{W} \times \mathrm{W} \mid \exists v \in \mathrm{W} \text{ s.t. } (w, v) \in \mathrm{R}_\sigma \text{ and } (v, u) \in \mathrm{R}_a\}$. Take a plan $\sigma \in \mathsf{Act}^*$: for $u \in \mathrm{W}$ define $\mathrm{R}_\sigma(u) := \{v \in \mathrm{W} \mid (u, v) \in \mathrm{R}_\sigma\}$, and for $U \subseteq \mathrm{W}$ define $\mathrm{R}_\sigma(U) := \bigcup_{u \in U} \mathrm{R}_\sigma(u)$. ⊣

[1] Thus, R_a might not be defined for some $a \in \mathsf{Act}$.

The idea in [58, 59, 60] is that an agent knows how to achieve φ given ψ when she has an appropriate plan that allows her to go from any state in which ψ holds only to states in which φ holds. A crucial part is, then, what "appropriate" is taken to be.

Definition 3.5 (Strong executability) Let $\{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in A, \text{ for some } A \subseteq \mathsf{Act}\}$ be a collection of binary relations. A plan $\sigma \in \mathsf{Act}^*$ is *strongly executable* (SE) at $u \in \mathrm{W}$ if and only if R_σ is defined and, additionally, $v \in \mathrm{R}_{\sigma_k}(u)$ implies $\mathrm{R}_{\sigma[k+1]}(v) \neq \varnothing$ for every $k \in [0 \,..\, |\sigma| - 1]$. We define the set $\mathrm{SE}(\sigma) := \{w \in \mathrm{W} \mid \sigma \text{ is SE at } w\}$. ⊣

Thus, strong executability asks for *every* partial execution of the plan (including ϵ) to be completed. With this notion, formulas in $\mathsf{L_{Kh}}$ are interpreted over an LTS as follows. Notice that the semantic clause for the Kh modality shown here is equivalent to the one found in the original papers.

Definition 3.6 ($\mathsf{L_{Kh}}$ over LTSs) The relation $\models$ between a pointed LTS $\mathcal{S}, w$ (with $\mathcal{S} = \langle \mathrm{W}, \mathrm{R}, \mathrm{V}, \mathsf{Act}\rangle$ an LTS and formulas in $\mathsf{L_{Kh}}$ over Prop) is defined inductively as follows:

$\mathcal{S}, w \models p$ *iff*$_{def}$ $p \in \mathrm{V}(w)$,

$\mathcal{S}, w \models \neg\varphi$ *iff*$_{def}$ $\mathcal{S}, w \not\models \varphi$,

$\mathcal{S}, w \models \varphi \vee \psi$ *iff*$_{def}$ $\mathcal{S}, w \models \varphi$ or $\mathcal{S}, w \models \psi$,

$\mathcal{S}, w \models \mathsf{Kh}(\psi, \varphi)$ *iff*$_{def}$ there exists $\sigma \in \mathsf{Act}^*$ such that **(Kh-*1*)** $[\![\psi]\!]^{\mathcal{S}} \subseteq \mathrm{SE}(\sigma)$ and **(Kh-*2*)** $\mathrm{R}_\sigma([\![\psi]\!]^{\mathcal{S}}) \subseteq [\![\varphi]\!]^{\mathcal{S}}$,

with $[\![\varphi]\!]^{\mathcal{S}} := \{w \in \mathrm{W} \mid \mathcal{S}, w \models \varphi\}$ (the elements in $[\![\varphi]\!]^{\mathcal{S}}$ are sometimes called φ-states). The plan σ in the semantic case for $\mathsf{Kh}(\psi, \varphi)$ is often called the witness for $\mathsf{Kh}(\psi, \varphi)$ in $\mathcal{S}$. ⊣

Thus, $\mathsf{Kh}(\psi, \varphi)$ holds at a given w when there is a plan σ such that, when it is executed at any ψ-state, it will always complete every partial execution (condition **(Kh-*1*)**), ending unerringly in states satisfying φ (condition **(Kh-*2*)**). Since w does not play any role in Kh's semantic clause, the *knowing how* operator acts *globally*. Hence, $[\![\mathsf{Kh}(\psi, \varphi)]\!]^{\mathcal{S}}$ is either $\mathrm{D}_{\mathcal{S}}$ or $\varnothing$.

Axiomatization. For axiomatization purposes, note that the global universal modality [20], interpreted as truth in every state of the model, is definable in $\mathsf{L_{Kh}}$ as $\mathsf{A}\varphi := \mathsf{Kh}(\neg\varphi, \bot)$. This is justified by the proposition below, whose proof relies on the fact that Act^* is never empty (it always contains ϵ).

Proposition 1 ([58]) *Let $\mathcal{S}, w$ be a pointed* LTS*. Then,*

$$\mathcal{S}, w \models \mathsf{Kh}(\neg\varphi, \bot) \quad \textit{iff} \quad [\![\varphi]\!]^{\mathcal{S}} = \mathrm{D}_{\mathcal{S}}. \qquad \triangleleft$$

The axiom system $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$ (Table 1) shows the relationship between the global universal modality A and the knowing-how operator Kh. The first block is essentially a standard modal system for A, additionally establishing that Kh is global (see the discussion in [58]). The axioms in the second block deserve a further comment. Axiom $\mathcal{EMP}$ states that, if $\psi \to \varphi$ is globally true, then given ψ the agent knows how to make φ true. In simpler words, global ontic information turns into knowledge. This is because the empty plan ϵ is always available. Axiom $\mathcal{COMP}$ establishes that Kh is compositional: if given ψ the

Block $\mathcal{L}$:	*TAUT*	$\vdash \varphi$ for φ a propositional tautology
	DISTA	$\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$
	TA	$\vdash \mathsf{A}\varphi \to \varphi$
	4KhA	$\vdash \mathsf{Kh}(\psi, \varphi) \to \mathsf{A}\mathsf{Kh}(\psi, \varphi)$
	5KhA	$\vdash \neg\mathsf{Kh}(\psi, \varphi) \to \mathsf{A}\neg\mathsf{Kh}(\psi, \varphi)$
	MP	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
	NECA	From $\vdash \varphi$ infer $\vdash \mathsf{A}\varphi$
Block $\mathcal{L}_{\text{LTS}}$:	*EMP*	$\vdash \mathsf{A}(\psi \to \varphi) \to \mathsf{Kh}(\psi, \varphi)$
	COMPKh	$\vdash (\mathsf{Kh}(\psi, \varphi) \wedge \mathsf{Kh}(\varphi, \chi)) \to \mathsf{Kh}(\psi, \chi)$

Table 1: Axiom system $\mathcal{L}_{\mathsf{Kh}}^{\text{LTS}}$, for L_{Kh} w.r.t. LTSs.

agent knows how to make φ true, and given φ she knows how to make χ true, then given ψ she knows how to make χ true.

Theorem 1 ([58]) *The axiom system $\mathcal{L}_{\mathsf{Kh}}^{\text{LTS}}$ (Table 1) is sound and strongly complete for L_{Kh} w.r.t. the class of all* LTS*s.* ◁

Axioms in the second block might be questionable. First, one could argue that, contrary to what *EMP* states, not all global truths about what is achievable in the model need to be considered as knowledge (how) of the agent. Second, notice that axiom *COMPKh* implies also a certain level of omniscience: it might as well be that an agent knows how to make φ true given ψ, and how to make χ true given φ, but still has not worked out how to put together the two witness plans to ensure χ given ψ. These are the two properties that will be lost in the more general semantics introduced in the next section. In Section 7 we will show how these formulas become valid when, in the new semantics, one make strong idealizations.

4 Uncertainty-based semantics

The LTS-based semantics provides a reasonable representation of an agent's abilities: the agent knows how to achieve φ given ψ if and only if there is a plan that, when executed at any ψ-state, will always complete every partial execution, ending unerringly in states satisfying φ. Still, one could argue that this representation involves a certain level of idealization.

Take an agent that *lacks* a certain ability. In the LTS-based semantics, this can only happen when the environment does not provide the required (sequence of) action(s). Still, there are situations in which an adequate plan exists, and yet the agent lacks the ability for a different reason. Indeed, she might *fail to distinguish* an adequate plan from a non-adequate one, in the sense of not being able to tell that, in general, those plans produce different outcomes. Consider, for example, an agent baking a cake. She might have the ability to do the nine different mixing methods[2] (beating, blending, creaming, cutting, folding,

[2] https://www.perfectlypastry.com/the-importance-of-the-mixing-method/

kneading, sifting, stirring, whipping), and she might even recognize them as different actions. However, she might not be able to perfectly distinguish one from the others: she might not recognize that, sometimes, they produce different results. In such cases, one would say that the agent does not know how to bake a cake: sometimes she gets good outcomes (when she uses the adequate mixing method) and sometimes she does not.

Indistinguishability among *basic* actions can account for the example above (with each mixing method a basic action). Still, one can also think of situations in which a more general form of indistinguishability, one *among plans*, is involved. Consider the baking agent again. It is reasonable to assume that she can tell the difference between "adding milk" and "adding flour", but perhaps she does not realize the effect that *the order* of these actions might have in the final result. Here, the issue is not that she cannot distinguish between basic actions; rather, two plans are indistinguishable because the order of their actions is being considered irrelevant. For a last possibility, the agent might not know that, while opening the oven once to check whether the baking goods are done is reasonable, this must not be done in excess. In this case, the problem consists in not being able to tell the difference between the effect of executing an action once and executing it multiple times. Thus, plans of *different lengths* might be considered equivalent for the task at hand, for such an agent.

The previous examples suggest that one can devise a more general representation of an agent's abilities. This involves taking into account not only the plans she has available (the LTS structure), but also her skills for telling two different plans apart (a form of *indistinguishability among plans*). As we will see, this (in)ability for distinguishing plans will also let us define a natural model for a multi-agent scenario. In this setting, agents share the same set of *affordances* (provided by the actual environment), but still have different *abilities*. This depends on how well they can tell these affordances apart, and on which of these affordances are available or not. To drive this last point home notice that, in principle, an agent does not need to have 'epistemic access' to every available plan. Some might be so foreign to the agent, or so complex, that she might not be *aware of* them. Such plans are, then, out of the agent's reach, not in the sense that she cannot distinguish them from others, but in that she does not even take them into consideration. This is similar to what [13] proposed for the epistemic notion of *knowing that*: the agent might not be aware of (i.e., she might not entertain) every formula of the language, and thus they are not part of her knowledge.

Definition 4.1 (Uncertainty-based LTS) Let Agt be a finite non-empty set of agents. A *multi-agent uncertainty-based* LTS (LTS^U) for Prop and Agt is a tuple $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \sim, \mathrm{V}, \mathsf{Act} \rangle$ where $\langle \mathrm{W}, \mathrm{R}, \mathrm{V}, \mathsf{Act} \rangle$ is an LTS and $\sim$ assigns, to each agent $i \in \mathsf{Agt}$, an equivalence *indistinguishability* relation over a non-empty set of plans $\mathrm{P}_i \subseteq \mathsf{Act}^*$. Given an LTS^U $\mathcal{M}$ and $w \in \mathrm{D}_{\mathcal{M}}$, the pair $(\mathcal{M}, w)$ (parenthesis usually dropped) is called a *pointed* LTS^U. ⊣

Intuitively, P_i is the set of plans that agent i has at her disposal; it contains the plans the agent has access to. Then, similarly as in classical epistemic logic, $\sim_i \subseteq \mathrm{P}_i \times \mathrm{P}_i$ describes agent i's indistinguishability over her available plans.

Remark 1 The following change in notation will simplify some definitions later on, and will make the comparison with the LTS-based semantics clearer.

Let $\langle W, R, \sim, V, \mathsf{Act}\rangle$ be an LTS^{U} and take $i \in \mathsf{Agt}$; for a plan $\sigma \in P_i$, let $[\sigma]_i$ be its equivalence class in $\sim_i$ (i.e., $[\sigma]_i := \{\sigma' \in P_i \mid \sigma \sim_i \sigma'\}$). There is a one-to-one correspondence between each $\sim_i$ and its induced set of equivalence classes $S_i := \{[\sigma]_i \mid \sigma \in P_i\}$. Hence, from now on, an LTS^{U} will be presented as a tuple $\langle W, R, \{S_i\}_{i\in\mathsf{Agt}}, V, \mathsf{Act}\rangle$. Notice the following properties of each S_i: ***(1)*** $S_i \neq \varnothing$ (as $P_i \neq \varnothing$), ***(2)*** if $\pi_1, \pi_2 \in S_i$ and $\pi_1 \neq \pi_2$, then $\pi_1 \cap \pi_2 = \varnothing$ (equivalence classes are pairwise disjoint), ***(3)*** $P_i = \bigcup_{\pi\in S_i} \pi$ (their union is exactly P_i), and ***(4)*** $\varnothing \notin S_i$ (the empty set is not an equivalence class). ⊣

Given her uncertainty over Act^* (or, more precisely, over *her* 'domain of plans' $P_i \subseteq \mathsf{Act}^*$), the abilities of an agent i depend not on what a single plan can achieve, but rather on what a set of them can guarantee.

Definition 4.2 For $\pi \subseteq \mathsf{Act}^*$, $u \in W$ and $U \subseteq W$, define

$$R_\pi := \bigcup_{\sigma\in\pi} R_\sigma, \qquad R_\pi(u) := \bigcup_{\sigma\in\pi} R_\sigma(u), \qquad R_\pi(U) := \bigcup_{u\in U} R_\pi(u).$$

⊣

We can now generalize the notion of strong executability for sets of plans.

Definition 4.3 (Strong executability) A *set of plans* $\pi \subseteq \mathsf{Act}^*$ is *strongly executable* at $u \in W$ if and only if *every* plan $\sigma \in \pi$ is *strongly executable* at u. Thus, $SE(\pi) := \bigcap_{\sigma\in\pi} SE(\sigma)$ is the set of the states in W where π is strongly executable. ⊣

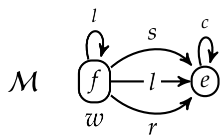
Definition 4.4 (Kh_i over LTS^{U}s) Let $\mathsf{L}_{\mathsf{Kh}_i}$ be the multi-agent version of the language L_{Kh}, obtained by replacing Kh with Kh_i (with $i \in \mathsf{Agt}$ for $\mathsf{Agt} \neq \varnothing$). Let $\mathcal{M} = \langle W, R, \{S_i\}_{i\in\mathsf{Agt}}, V, \mathsf{Act}\rangle$ be an LTS^{U} over Prop and Agt and let $w \in W$. The satisfiability relation $\models$ between $\mathcal{M}, w$ and formulas in $\mathsf{L}_{\mathsf{Kh}_i}$ is defined inductively. The atomic and Boolean cases are as before. For *knowing how* formulas,

$\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi)$ *iff*$_{def}$ there exists $\pi \in S_i$ such that
(Kh-1) $[\![\psi]\!]^{\mathcal{M}} \subseteq SE(\pi)$ and ***(Kh-2)*** $R_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$,

with $[\![\varphi]\!]^{\mathcal{M}} := \{w \in W \mid \mathcal{M}, w \models \varphi\}$. The set of plans π in the semantic clause for $\mathsf{Kh}_i(\psi, \varphi)$ is often called the witness for $\mathsf{Kh}_i(\psi, \varphi)$ in $\mathcal{M}$. ⊣

It is worth comparing Definition 3.6 and Definition 4.4. As before, $\mathsf{Kh}_i(\psi, \varphi)$ acts *globally*. But now, we require *for agent i* to have a *set of plans* satisfying strong executability in every ψ-state (condition ***(Kh-1)***). Still, the set of plans should work as the single plan did before: when executed at ψ-states, it should end unerringly in states satisfying φ (condition ***(Kh-2)***). Below we provide an example to illustrate the notions just introduced.

Example 1 Let us consider an evacuation protocol of a given building. Naturally, there exist certain courses of action that, in the case of an emergency, safely lead to the evacuation point, while others may fail. For instance, if a fire emergency (f) occurs, there are in this case three possible exit routes: using the stairs (action s), using the ramp (action r), or using the lift (action l). While using the stairs or the ramp guarantee that the agents reach the evacuation point (e), this is not true for the lift. The evacuation protocol indicates that in the case of a fire, the agents first must take the stairs or the ramp, and finally call 911 (c). Thus, the two plans to use in case of fire are sc and rc. While taking the lift at any point is disallowed. This situation is illustrated in the picture below.



$$\mathsf{S}_i = \left\{ \ \{sc, rc\}, \{lc\} \ \right\} \qquad \mathsf{S}_j = \left\{ \ \{lc, sc, rc\} \ \right\}$$

To the right of the LTS $\mathcal{M}$, the sets S_i and S_j show the uncertainty sets of agents i and j. Notice that both agents are aware of the existence of the three possible courses of action, i.e., $\{sc, rc, lc\} \subseteq \mathrm{P}_i \cap \mathrm{P}_j$. Agent i, who has taken emergency training, considers that sc and rc are equally good evacuation strategies in case of fire, while lc is not. Indeed, following Definition 4.4, we have $\mathcal{M}, w \models \mathsf{Kh}_i(f, e)$, i.e., agent i knows how to achieve e given f, as the set $\{sc, rc\}$ acts as a witness for the formula. On the other hand, $\mathcal{M}, w \not\models \mathsf{Kh}_j(f, e)$: agent j has only one equivalence class, and one of its plans (lc) is not strongly executable given f. ⊣

In our semantics, we have not imposed any restrictions on the uncertainty sets S_i. One may wonder whether some restrictions should be imposed on them or not. For instance whether $a \sim_i b$ must imply $ab \sim_j ba$. This property states that atomic indistinguishability implies indistinguishability at the level of more complex plans. Still, consider the following situation, which shows that this property does not necessarily hold in general. Agent i usually gets somewhat hungry in the afternoon, and she usually eats a small snack, sometimes it is sweet, sometimes it is savory, she has no real preference between the two (i.e., $a \sim_i b$). When she is hungrier, she usually has both (a savory and a sweet snack). But in those cases, she will first have the savory and then the sweet one (as a dessert). She would never do it the other way round (i.e., it is not the case that $ab \sim_i ba$). Being said that, in this paper, we aim to characterize a base logic for knowing how, it seems correct to impose only the minimal conditions required of the indistinguishability relation.

It is also important to notice that the global universal modality is also definable within $\mathsf{L}_{\mathsf{Kh}_i}$ over LTS^U. (For this, it is crucial that $\mathsf{S}_i \neq \varnothing$ and $\varnothing \notin \mathsf{S}_i$, as stated in Remark 1.)

Proposition 2 *Let $\mathcal{M}, w$ be a pointed LTS^U. Then,*

$$\textit{there is } i \in \mathsf{Agt} \textit{ with } \mathcal{M}, w \models \mathsf{Kh}_i(\neg\varphi, \bot) \qquad \textit{iff} \qquad [\![\varphi]\!]^{\mathcal{M}} = \mathrm{D}_{\mathcal{M}}.$$

Proof. **(⇒)** Suppose there is $i \in \mathsf{Agt}$ with $\mathcal{M}, w \models \mathsf{Kh}_i(\neg\varphi, \bot)$. Then, there is $\pi \in \mathsf{S}_i$ such that ***(Kh-1)*** $[\![\neg\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and ***(Kh-2)*** $\mathrm{R}_\pi([\![\neg\varphi]\!]^{\mathcal{M}}) \subseteq [\![\bot]\!]^{\mathcal{M}}$. For a contradiction, suppose $[\![\varphi]\!]^{\mathcal{M}} \neq \mathrm{D}_{\mathcal{M}}$, so there is $u \in [\![\neg\varphi]\!]^{\mathcal{M}}$. Then, Item ***(Kh-1)*** implies $u \in \mathrm{SE}(\pi) = \bigcap_{\sigma\in\pi} \mathrm{SE}(\sigma)$. But $\pi \in \mathsf{S}_i$, so $\pi \neq \varnothing$, that is, there is $\sigma \in \pi$ with $u \in \mathrm{SE}(\sigma)$; thus, $\mathrm{R}_\sigma(u) \neq \varnothing$, so $\mathrm{R}_\pi(u) \neq \varnothing$ and hence $\mathrm{R}_\pi([\![\neg\varphi]\!]^{\mathcal{M}}) \neq \varnothing$, that is, $\varnothing \subset \mathrm{R}_\pi([\![\neg\varphi]\!]^{\mathcal{M}})$. But then, from Item ***(Kh-2)***, $\varnothing \subset \mathrm{R}_\pi([\![\neg\varphi]\!]^{\mathcal{M}}) \subseteq [\![\bot]\!]^{\mathcal{M}}$, i.e., $\varnothing \subset [\![\bot]\!]^{\mathcal{M}}$, a contradiction. Therefore, $[\![\varphi]\!]^{\mathcal{M}} = \mathrm{D}_{\mathcal{M}}$.

(⇐) Suppose $[\![\varphi]\!]^{\mathcal{M}} = \mathrm{D}_{\mathcal{M}}$. Then $[\![\neg\varphi]\!]^{\mathcal{M}} = \varnothing$ and hence ***(Kh-1)*** in the semantic clause of $\mathsf{Kh}_i(\neg\varphi, \bot)$ holds for every $\pi \in 2^{(\mathsf{Act}^*)}$. Moreover, $\mathrm{R}_\pi([\![\neg\varphi]\!]^{\mathcal{M}}) = \bigcup_{u\in[\![\neg\varphi]\!]^{\mathcal{M}}} \mathrm{R}_\pi(u) = \bigcup_{u\in\varnothing} \mathrm{R}_\pi(u) = \varnothing$, so ***(Kh-2)*** also holds for any such π. Finally, $\mathsf{S}_i \neq \varnothing$ (so there is $\pi \in \mathsf{S}_i$) and $\mathsf{Agt} \neq \varnothing$ (so there is $i \in \mathsf{Agt}$); therefore, there is $i \in \mathsf{Agt}$ with $\mathcal{M}, w \models \mathsf{Kh}_i(\neg\varphi, \bot)$. ∎

Hence, one can take $\mathsf{A}\varphi := \bigvee_{i\in\mathsf{Agt}} \mathsf{Kh}_i(\neg\varphi, \bot)$ (recall: Agt is non-empty and finite) and $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$.

Now, clearly different agents have different awareness about their own abilities. At the same time, because of the global nature of the modality of knowing how, it holds that

$$\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi) \text{ if and only if } \mathcal{M}, w \models \mathsf{AKh}_i(\psi, \varphi),$$

or equivalenty,

$$\mathcal{M}, w \models \mathsf{Kh}_j(\neg \mathsf{Kh}_i(\psi, \varphi), \bot), \text{ for some agent } j.$$

But this does not imply that agent j *knows that* "agent i knows how to achive φ given ψ". It is only the case that $\mathsf{Kh}_i(\psi, \varphi)$ becomes an objective true, and hence assuming its negation naturally leads to a contradiction. There is no notion of epistemic indistinguishability over states in our models, which could lead to a notion of "knows that".

Lastly, one can argue that since models are equipped with a notion of epistemic indistinguishability between plans, an agent should know that a certain plan is (or is not) distinguishable from another, or that an agent is aware of the availability of a certain course of action. However, knowing how modalities cannot talk about the relation itself, only about the existence of a set of indistinguishable plans, and the effects of executing those plans.

5 Bisimulations

Bisimulation is a crucial tool for understanding the expressive power of a formal language. In [18, 19], bisimulation notions for L_{Kh} over LTSs have been introduced. This section discusses similar ideas for $\mathsf{L}_{\mathsf{Kh}_i}$ over LTS^Us.

First, a useful abbreviation.

Definition 5.1 Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathrm{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V}, \mathsf{Act} \rangle$ be an LTS^U over Prop and Agt. Take a set of plans $\pi \in 2^{(\mathsf{Act}^*)}$, sets of states $U, T \subseteq \mathrm{W}$ and an agent $i \in \mathsf{Agt}$.

- Write $U \overset{\pi}{\Rrightarrow} T$ *iff*$_{def}$ $U \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi(U) \subseteq T$.
- Write $U \overset{i}{\Rrightarrow} T$ *iff*$_{def}$ there is $\pi \in \mathrm{S}_i$ such that $U \overset{\pi}{\Rrightarrow} T$.

Additionally, $U \subseteq \mathrm{W}$ is $\mathsf{L}_{\mathsf{Kh}_i}$-definable (respectively, propositionally definable) in $\mathcal{M}$ if and only if there is an $\mathsf{L}_{\mathsf{Kh}_i}$-formula (propositional formula) φ such that $U = [\![\varphi]\!]^{\mathcal{M}}$. ⊣

Two quick observations. First, note how the abbreviation simplifies the semantic clause for *knowing how* formulas: $\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi)$ if and only if $[\![\psi]\!]^{\mathcal{M}} \overset{i}{\Rrightarrow} [\![\varphi]\!]^{\mathcal{M}}$. Second, under the LTS^U-based semantics, $\mathsf{L}_{\mathsf{Kh}_i}$-definability implies propositional definability. Its proof, analogous to the LTS-based semantics case in [18, 19], relies on the fact that Kh_i acts globally.

Proposition 3 *Let $\mathcal{M}$ be an LTS^U. For all $U \subseteq \mathrm{D}_{\mathcal{M}}$, if U is $\mathsf{L}_{\mathsf{Kh}_i}$-definable, then it is propositionally definable.* ◁

We now introduce the notion of bisimulation. Note that, although the collection of binary relations of a model is not explicitly mentioned, it is referred to through the abstract relation "$\overset{i}{\Rrightarrow}$" (Definition 5.1).

Definition 5.2 ($\mathsf{L_{Kh_i}}$-bisimulation) Let $\mathcal{M}$ and $\mathcal{M}'$ be two LTS^{U}s, their domains being W and W′, respectively. Take $Z \subseteq W \times W'$.

- For $u \in W$ and $U \subseteq W$, define

$$Z(u) := \{u' \in W' \mid uZu'\}, \qquad Z(U) := \bigcup_{u \in U} Z(u).$$

- For $u' \in W'$ and $U' \subseteq W'$, define

$$Z^{-1}(u') := \{u \in W \mid uZu'\}; \qquad Z^{-1}(U') := \bigcup_{u' \in U'} Z^{-1}(u').$$

A non-empty $Z \subseteq W \times W'$ is called an $\mathsf{L_{Kh_i}}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if and only if wZw' implies all of the following.

- **Atom**: $V(w) = V'(w')$.
- Kh_i**-Zig**: for any *propositionally* definable $U \subseteq W$, if $U \overset{i}{\Rrightarrow} T$ for some $T \subseteq W$, then there is $T' \subseteq W'$ satisfying both

  (B1) $Z(U) \overset{i}{\Rrightarrow} T'$, ***(B2)*** $T' \subseteq Z(T)$.

- Kh_i**-Zag**: for any *propositionally* definable $U' \subseteq W'$, if $U' \overset{i}{\Rrightarrow} T'$ for some $T' \subseteq W'$, then there is $T \subseteq W$ satisfying both

  (B1) $Z^{-1}(U') \overset{i}{\Rrightarrow} T$, ***(B2)*** $T \subseteq Z^{-1}(T')$.

- **A-Zig**: for all u in W there is a u' in W′ such that uZu'.
- **A-Zag**: for all u' in W′ there is a u in W such that uZu'.

We write $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ when there is an $\mathsf{L_{Kh}}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that wZw'. ⊣

The two requirements in Kh_i-Zig are equivalent to a single one: $Z(U) \overset{i}{\Rrightarrow} Z(T)$. They are split to resemble more closely the definition of a standard bisimulation: if U has an 'i-successor' T, then its 'bisimulation image' $Z(U)$ also has an 'i-successor', namely T' (clause $Z(U) \overset{i}{\Rrightarrow} T'$), and these successors are a 'bisimilar match' (clause $T' \subseteq Z(T)$). The case of Kh_i-Zag is analogous.

In order to formalize the crucial properties of a bisimulation, we define the notion of model equivalence with respect to $\mathsf{L_{Kh_i}}$.

Definition 5.3 ($\mathsf{L_{Kh_i}}$-equivalence) Two pointed LTS^{U}s $\mathcal{M}, w$ and $\mathcal{M}', w'$ are $\mathsf{L_{Kh_i}}$*-equivalent* (written $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$) if and only if, for every $\varphi \in \mathsf{L_{Kh_i}}$,

$$\mathcal{M}, w \models \varphi \quad \text{iff} \quad \mathcal{M}', w' \models \varphi. \qquad ⊣$$

Then, we can state the intended correspondence between $\leftrightarroweq$ and $\leftrightsquigarrow$.

Theorem 2 ($\mathsf{L_{Kh_i}}$-bisimilarity implies $\mathsf{L_{Kh_i}}$-equivalence) *Let $\mathcal{M}, w$ and $\mathcal{M}', w'$ be pointed* LTS^U*s. Then,*

$$\mathcal{M}, w \leftrightarroweq \mathcal{M}', w' \quad \textit{implies} \quad \mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'.$$

Proof. Take $\mathcal{M} = \langle W, R, \{S_i\}_{i \in \mathsf{Agt}}, V, \mathsf{Act} \rangle$ and $\mathcal{M}' = \langle W', R', \{S'_i\}_{i \in \mathsf{Agt}}, V', \mathsf{Act}' \rangle$. From the given $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$, there is an $\mathsf{L_{Kh_i}}$-bisimulation $Z \subseteq (W \times W')$ with wZw'. The proof of $\mathsf{L_{Kh_i}}$-equivalence is by structural induction on $\mathsf{L_{Kh_i}}$-formulas. The cases for atomic propositions and Boolean operators are standard, and only formulas of the form $\mathsf{Kh}_i(\psi, \varphi)$ are left. Note how, for this case, the inductive hypothesis (IH) states that, for $u \in W$, $u' \in W'$ and χ a subformula of $\mathsf{Kh}_i(\psi, \varphi)$, if uZu' then $u \in [\![\chi]\!]^{\mathcal{M}}$ iff $u' \in [\![\chi]\!]^{\mathcal{M}'}$.

Suppose $w \in [\![\mathsf{Kh}_i(\psi, \varphi)]\!]^{\mathcal{M}}$. Then, by semantic interpretation, $[\![\psi]\!]^{\mathcal{M}} \xRightarrow{i} [\![\varphi]\!]^{\mathcal{M}}$. It is useful to notice that $Z([\![\chi]\!]^{\mathcal{M}}) = [\![\chi]\!]^{\mathcal{M}'}$ holds for $\chi \in \{\psi, \varphi\}$.

(⊆) If $v' \in Z([\![\chi]\!]^{\mathcal{M}})$, then there is $v \in [\![\chi]\!]^{\mathcal{M}}$ such that vZv'. Thus, from IH we have $v' \in [\![\chi]\!]^{\mathcal{M}'}$.

(⊇) If $v' \in [\![\chi]\!]^{\mathcal{M}'}$ then, by A-Zag, there is v with vZv'; thus, from IH we have $v \in [\![\chi]\!]^{\mathcal{M}}$. Hence, $v' \in Z([\![\chi]\!]^{\mathcal{M}})$.

Now, the proof. We have $[\![\psi]\!]^{\mathcal{M}} \xRightarrow{i} [\![\varphi]\!]^{\mathcal{M}}$. The set $[\![\psi]\!]^{\mathcal{M}}$ is obviously $\mathsf{L}_{\mathsf{Kh}_i}$-definable, and hence propositionally definable too (Proposition 3). Then, from the Kh_i-Zig clause, there is $T' \subseteq W'$ such that ***(B1)*** $Z([\![\psi]\!]^{\mathcal{M}}) \xRightarrow{i} T'$ and ***(B2)*** $T' \subseteq Z([\![\varphi]\!]^{\mathcal{M}})$. Therefore, $Z([\![\psi]\!]^{\mathcal{M}}) \xRightarrow{i} Z([\![\varphi]\!]^{\mathcal{M}})$ and hence, by the result above, $[\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} [\![\varphi]\!]^{\mathcal{M}'}$. Thus, $w' \in [\![\mathsf{Kh}_i(\psi, \varphi)]\!]^{\mathcal{M}'}$.

The direction from $w' \in [\![\mathsf{Kh}_i(\psi, \varphi)]\!]^{\mathcal{M}'}$ to $w \in [\![\mathsf{Kh}_i(\psi, \varphi)]\!]^{\mathcal{M}}$ follows a similar argument, using A-Zig and Kh_i-Zag instead. ∎

It is easy to see that the converse of Theorem 2 does not hold over arbitrary models: in fact, the counterexample provided in [19, Section 2] serves also here to make our point. To satisfy the converse, we usually need to restrict ourselves to some particular classes of models, that are in general known as Hennessy-Milner classes. In many modal logics, one typically works only with image-finite models: those in which, at every state, every basic relation has only finitely many successors. For languages in which the global universal modality A is definable, as $\mathsf{L}_{\mathsf{Kh}_i}$, this requirement needs to be strengthened, as every state can reach (via the relation underlying A) every other state. For instance, we can take the class of models with a finite domain. Precisely, here we show that the class of finite models (taken as those with a finite domain) forms a Hennessy-Milner class. Note that we do not impose any restriction over the uncertainty relation of the models (or the sets S_i).

Theorem 3 ($\mathsf{L}_{\mathsf{Kh}_i}$-equivalence implies $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilarity) *Let $\mathcal{M}, w$ and $\mathcal{M}', w'$ be* finite *pointed* LTS^U*s. Then,*

$$\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w' \quad \textit{implies} \quad \mathcal{M}, w \leftrightarrows \mathcal{M}', w'.$$

Proof. Take $\mathcal{M} = \langle W, R, \{S_i\}_{i \in \mathsf{Agt}}, V, \mathsf{Act}\rangle$ and $\mathcal{M}' = \langle W', R', \{S'_i\}_{i \in \mathsf{Agt}}, V', \mathsf{Act}'\rangle$. The strategy is to show that the relation $\leftrightsquigarrow$ is already a $\mathsf{L}_{\mathsf{Kh}_i}$–bisimulation. Thus, define

$$Z := \{(v, v') \in (W \times W') \mid \mathcal{M}, v \leftrightsquigarrow \mathcal{M}', v'\}$$

so wZw' implies w and w' satisfy exactly the same $\mathsf{L}_{\mathsf{Kh}_i}$-formulas. In order to show that Z satisfies the requirements, take any $(w, w') \in Z$.

- **Atom**. States w and w' agree in all $\mathsf{L}_{\mathsf{Kh}_i}$-formulas, and thus in all atoms.
- **A-Zig**. Take $v \in W$ and suppose, for the sake of a contradiction, that there is no $v' \in W'$ such that vZv'. Then, from Z's definition, for each $v'_i \in W' = \{v'_1, \ldots, v'_n\}$ (recall: $\mathcal{M}'$ is finite) there is an L_{Kh}-formula θ_i such that $\mathcal{M}, v \models \theta_i$ but $\mathcal{M}', v'_i \not\models \theta_i$. Now take $\theta := \theta_1 \wedge \cdots \wedge \theta_n$. Clearly, $\mathcal{M}, v \models \theta$; however, $\mathcal{M}', v'_i \not\models \theta$ for each $v'_i \in W'$, as each one of them makes 'its' conjunct θ_i false. Then, $\mathcal{M}, w \models \mathsf{E}\theta$ but $\mathcal{M}', w' \not\models \mathsf{E}\theta$, contradicting the assumption wZw'.

- A**-Zag**. Analogous to the A-Zig case.
- Kh_i**-Zig**. Take any propositionally definable set $[\![\psi]\!]^{\mathcal{M}} \subseteq \mathrm{W}$ (thus, ψ is propositional), and suppose $[\![\psi]\!]^{\mathcal{M}} \xRightarrow{i} T$ for some $T \subseteq \mathrm{W}$. We need to find a $T' \subseteq \mathrm{W}'$ satisfying both

  (B1) $Z([\![\psi]\!]^{\mathcal{M}}) \xRightarrow{i} T'$, ***(B2)*** $T' \subseteq Z(T)$.

  Note that $Z([\![\psi]\!]^{\mathcal{M}}) = [\![\psi]\!]^{\mathcal{M}'}$. For **(⊇)**, suppose $u' \in [\![\psi]\!]^{\mathcal{M}'}$. From A-Zag (proved above), there is $u \in \mathrm{W}$ such that uZu'; then, from Z's definition, $u \in [\![\psi]\!]^{\mathcal{M}}$ so $u' \in Z([\![\psi]\!]^{\mathcal{M}})$. For **(⊆)**, suppose $u' \in Z([\![\psi]\!]^{\mathcal{M}})$. Then, there is $u \in [\![\psi]\!]^{\mathcal{M}}$ such that uZu', and therefore, from Z's definition, $u' \in [\![\psi]\!]^{\mathcal{M}'}$. Thus, we actually require a $T' \subseteq \mathrm{W}'$ satisfying both

  (B1) $[\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'$, ***(B2)*** $T' \subseteq Z(T)$.

  Now, consider two alternatives.

  (1) Assume $[\![\psi]\!]^{\mathcal{M}} = \varnothing$. Then, $[\![\psi]\!]^{\mathcal{M}'} = Z([\![\psi]\!]^{\mathcal{M}}) = \varnothing$ and hence $T' = \varnothing$ does the job, as the following hold

  (B1) $\varnothing \xRightarrow{i} \varnothing$ (as $\mathsf{S}_i \neq \varnothing$), ***(B2)*** $\varnothing \subseteq Z(T)$.

  (2) Assume $[\![\psi]\!]^{\mathcal{M}} \neq \varnothing$. This gives us $T \neq \varnothing$ (from $[\![\psi]\!]^{\mathcal{M}} \xRightarrow{i} T$), which will be useful later. To show that there is a $T' \subseteq \mathrm{W}'$ satisfying both ***(B1)*** and ***(B2)***, we proceed by contradiction, so suppose there is no T' satisfying both requirements: every $T' \subseteq \mathrm{W}'$ satisfying ***(B1)*** fails at ***(B2)***. In other words, every $T' \subseteq \mathrm{W}'$ satisfying $[\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'$ has a state $v'_{T'} \in T'$ that is not the Z-image of some state $v \in T$ (i.e., $vZv'_{T'}$ fails for every $v \in T$). From Z's definition, the latter means that every state in T can be distinguished from this $v'_{T'}$ by an $\mathsf{L}_{\mathsf{Kh}_i}$-formula. Thus, given any $T' \subseteq \mathrm{W}'$ with $[\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'$, one can find a state $v'_{T'} \in T'$ such that, for each $v \in T$, there is a formula $\theta^{v}_{v'_{T'}}$ with $\mathcal{M}, v \models \theta^{v}_{v'_{T'}}$ but $\mathcal{M}', v'_{T'} \not\models \theta^{v}_{v'_{T'}}$. Then, for each such $v'_{T'}$ in each such T' define

  $$\theta_{T'} := \bigvee_{v \in T} \theta^{v}_{v'_{T'}} \qquad \text{and then} \qquad \theta := \bigwedge_{\{T' \subseteq \mathrm{W}' \mid [\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'\}} \theta_{T'},$$

  Observe the following. First, $\theta_{T'}$ is indeed a formula, as W is finite and thus so is T. Equally important, $T \neq \varnothing$, and thus $\theta_{T'}$ does not collapse to $\bot$. Second, θ is also a formula, as W' is finite and thus so is $\{T' \subseteq \mathrm{W}' \mid [\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'\}$. However, the latter set might be empty. This is what creates the following two cases.

  - Suppose $\{T' \subseteq \mathrm{W}' \mid [\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'\} = \varnothing$. Then, consider the formula $\mathsf{Kh}_i(\psi, \top)$. Since $[\![\psi]\!]^{\mathcal{M}} \xRightarrow{i} T$ and $T \subseteq \mathrm{W} = [\![\top]\!]^{\mathcal{M}}$, it follows that $\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \top)$. However, $\mathcal{M}', w' \not\models \mathsf{Kh}_i(\psi, \top)$ as, according to this case, there is no $T' \subseteq \mathrm{W}'$ with $[\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'$. This contradicts the $\mathsf{L}_{\mathsf{Kh}_i}$-equivalence of w and w'.
  - Suppose $\{T' \subseteq \mathrm{W}' \mid [\![\psi]\!]^{\mathcal{M}'} \xRightarrow{i} T'\} \neq \varnothing$. Then, θ does not collapse to $\top$. Now, note how every $v \in T$ satisfies its 'own' disjunct $\theta^{v}_{v'_{T'}}$ in each conjunct $\theta_{T'}$, and thus it satisfies θ. Thus, $T \subseteq [\![\theta]\!]^{\mathcal{M}}$

Block $\mathcal{L}$:	$\mathcal{TAUT}$	$\vdash \varphi$ for φ a propositional tautology
	$\mathcal{DISTA}$	$\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$
	$\mathcal{TA}$	$\vdash \mathsf{A}\varphi \to \varphi$
	$\mathcal{4KhA}$	$\vdash \mathsf{Kh}_i(\psi,\varphi) \to \mathsf{A}\mathsf{Kh}_i(\psi,\varphi)$
	$\mathcal{5KhA}$	$\vdash \neg\mathsf{Kh}_i(\psi,\varphi) \to \mathsf{A}\neg\mathsf{Kh}_i(\psi,\varphi)$
	$\mathcal{MP}$	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
	$\mathcal{NECA}$	From $\vdash \varphi$ infer $\vdash \mathsf{A}\varphi$
Block $\mathcal{L}_{\mathrm{LTS}^U}$:	$\mathcal{KhE}$	$\vdash (\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\varphi)) \to \mathsf{E}\varphi$
	$\mathcal{KhA}$	$\vdash (\mathsf{A}(\chi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \theta)) \to \mathsf{Kh}_i(\chi,\theta)$

Table 2: Axioms $\mathcal{L}_{\mathrm{LTS}^U}$, for $\mathsf{L}_{\mathsf{Kh}_i}$ w.r.t. $\mathrm{LTS}^U s$.

and hence, from $[\![\psi]\!]^{\mathcal{M}} \overset{i}{\Rightarrow} T$ and the fact that Kh_i-formulas are global, it follows that $\mathcal{M}, w \models \mathsf{Kh}_i(\psi,\theta)$. However, for each T' in $\{T' \subseteq \mathrm{W}' \mid [\![\psi]\!]^{\mathcal{M}'} \overset{i}{\Rightarrow} T'\}$, the state $v'_{T'}$ that cannot be matched with any state $v \in T$ makes all disjuncts in $\theta_{T'}$ false, thus falsifying $\theta_{T'}$ and therefore falsifying θ too. In other words, every $T' \subseteq \mathrm{W}'$ with $[\![\psi]\!]^{\mathcal{M}'} \overset{i}{\Rightarrow} T'$ contains a state $t'_{T'}$ with $\mathcal{M}', t'_{T'} \not\models \theta$, that is, $[\![\psi]\!]^{\mathcal{M}'} \overset{i}{\Rightarrow} T'$ implies $T' \not\subseteq [\![\theta]\!]^{\mathcal{M}}$. Hence, using again the fact that Kh_i-formulas are global, $\mathcal{M}', w' \not\models \mathsf{Kh}_i(\psi,\theta)$, contradicting the $\mathsf{L}_{\mathsf{Kh}_i}$-equivalence of w and w'.

- Kh_i**-Zag**. Analogous to the Kh_i-Zig case. ■

6 Axiomatization

We now present a sound and complete axiom system for $\mathsf{L}_{\mathsf{Kh}_i}$ under the LTS^U-based semantics. Recall that $\mathsf{A}\varphi := \bigvee_{i \in \mathsf{Agt}} \mathsf{Kh}_i(\neg\varphi, \bot)$ and $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$. With this, it turns out that formulas and rules in $\mathcal{L}$ (the first block of Table 1) are still sound under LTS^U (provided Kh is replaced by Kh_i). They will constitute the first part of an axiom system for $\mathsf{L}_{\mathsf{Kh}_i}$ over LTS^U (first block in Table 2). Still, this is not enough for a complete axiom system. The axioms on the second block of Table 2, $\mathcal{KhE}$ and $\mathcal{KhA}$, are the missing pieces[3].

We should start by discussing the newly introduced axioms. On the one hand, axiom $\mathcal{KhA}$ can be subjected to some of the criticisms that apply to $\mathcal{EMP}$ and $\mathcal{COMPKh}$ but, in our opinion, to a lesser extent. It implies certain level of idealization, as it entails that the abilities of the agent are, in a sense, closed under global entailment. On the other hand, axiom $\mathcal{KhE}$ states a simple reasonable requirement: if $\mathsf{Kh}_i(\psi,\varphi)$ is not trivial (given that $\mathsf{E}\psi$ holds), then $\mathsf{E}\varphi$ should be assured.

[3] $\mathcal{KhE}$ and $\mathcal{KhA}$ are also valid under LTS semantics (Proposition 10). In fact, the next section will show that $\mathcal{L}^{\mathrm{LTS}^U}_{\mathsf{Kh}_i}$ (for LTS^U) is strictly weaker than $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$ (for LTS).

Define the system $\mathcal{L}_{\mathsf{Kh}_i}^{\mathrm{LTS}^U} := \mathcal{L}$ (first block of Table 2) + $\mathcal{L}_{\mathrm{LTS}^U}$ (second block of Table 2). Now we will show that $\mathcal{L}_{\mathsf{Kh}_i}^{\mathrm{LTS}^U}$ is sound and strongly complete for $\mathsf{L}_{\mathsf{Kh}_i}$ over LTS^Us. Proving soundness is rather straightforward, so we will focus on strong completeness. Following [10, Proposition 4.12], the strategy is to build, for any $\mathcal{L}_{\mathsf{Kh}_i}^{\mathrm{LTS}^U}$-consistent set of formulas, an LTS^U satisfying them. In particular, the notions of theoremhood, (local) consequence, inconsistency, and maximally consistent sets are defined as usual [10]. We will rely on ideas from [58, 60]; the following theorems will be useful.

Proposition 4 *Formulas* $\mathsf{A}\neg\psi \to \mathsf{Kh}_i(\psi,\varphi)$ *(called* $\mathcal{SCOND}$*) and* $\mathsf{Kh}_i(\bot,\varphi)$ *(called* $\mathcal{COND}$*) are* $\mathcal{L}_{\mathsf{Kh}_i}^{\mathrm{LTS}^U}$*-derivable. That is,* **(1)** $\vdash \mathsf{A}\neg\psi \to \mathsf{Kh}_i(\psi,\varphi)$ *and* **(2)** $\vdash \mathsf{Kh}_i(\bot,\varphi)$.

Proof. ***(1)*** Take $\vdash \mathsf{A}(\psi \to \psi) \to (\mathsf{A}(\bot \to \varphi) \to (\mathsf{Kh}_i(\psi,\bot) \to \mathsf{Kh}_i(\psi,\varphi)))$, an instance of $\mathcal{KhA}$. Using $\mathcal{TAUT}$ and $\mathcal{NECA}$ we get $\vdash \mathsf{A}(\psi \to \psi)$; analogously, we get $\vdash \mathsf{A}(\bot \to \varphi)$. Then, using $\mathcal{MP}$ twice yields $\vdash \mathsf{Kh}_i(\psi,\bot) \to \mathsf{Kh}_i(\psi,\varphi)$, which by A's definition is $\vdash \mathsf{A}\neg\psi \to \mathsf{Kh}_i(\psi,\varphi)$. ***(2)*** Take $\vdash \mathsf{A}\neg\bot \to \mathsf{Kh}_i(\bot,\varphi)$, an instance of the previous item. Using $\mathcal{TAUT}$ and $\mathcal{NECA}$ we get $\vdash \mathsf{A}\neg\bot$ so, by $\mathcal{MP}$, $\vdash \mathsf{Kh}_i(\bot,\varphi)$. ■

Here it is, then, the definition of the required LTS^U.

Definition 6.1 Let $\mathbf{\Phi}$ be the set of all maximally $\mathcal{L}_{\mathsf{Kh}_i}^{\mathrm{LTS}^U}$-consistent sets (MCS) of formulas in $\mathsf{L}_{\mathsf{Kh}_i}$. For any $\Delta \in \mathbf{\Phi}$, define

$$\Delta|_{\mathsf{Kh}_i} := \{\xi \in \Delta \mid \xi \text{ is of the form } \mathsf{Kh}_i(\psi,\varphi)\}, \qquad \Delta|_{\mathsf{Kh}} := \bigcup_{i\in\mathsf{Agt}} \Delta|_{\mathsf{Kh}_i}.$$
$$\Delta|_{\neg\mathsf{Kh}_i} := \{\xi \in \Delta \mid \xi \text{ is of the form } \neg\mathsf{Kh}_i(\psi,\varphi)\}, \qquad \Delta|_{\neg\mathsf{Kh}} := \bigcup_{i\in\mathsf{Agt}} \Delta|_{\neg\mathsf{Kh}_i}.$$

Let Γ be a set in $\mathbf{\Phi}$; we will define a structure satisfying its formulas. Define a set of basic actions $\mathsf{Act}_i^\Gamma := \{\langle\psi,\varphi\rangle \mid \mathsf{Kh}_i(\psi,\varphi) \in \Gamma\}$ associated to each agent $i \in \mathsf{Agt}$, and then their union $\mathsf{Act}^\Gamma := \bigcup_{i\in\mathsf{Agt}} \mathsf{Act}_i^\Gamma$. Notice that $\mathsf{Kh}_i(\bot,\varphi) \in \Gamma$ for every $i \in \mathsf{Agt}$ and every $\varphi \in \mathsf{L}_{\mathsf{Kh}_i}$ (by $\mathcal{COND}$); since Agt is finite and non-empty, this implies that Act^Γ is enumerable, and thus it is an adequate set of actions for building a model.

Then, the structure $\mathcal{M}^\Gamma = \langle \mathrm{W}^\Gamma, \mathrm{R}^\Gamma, \{\mathrm{S}_i^\Gamma\}_{i\in\mathsf{Agt}}, \mathrm{V}^\Gamma, \mathsf{Act}^\Gamma\rangle$, over Agt and Prop is defined as follows.

- $\mathrm{W}^\Gamma := \{\Delta \in \mathbf{\Phi} \mid \Delta|_{\mathsf{Kh}} = \Gamma|_{\mathsf{Kh}}\}$.
- $\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle} := \bigcup_{i\in\mathsf{Agt}} \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle^i}$, with
$$\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle^i} := \{(\Delta_1,\Delta_2) \in \mathrm{W}^\Gamma \times \mathrm{W}^\Gamma \mid \mathsf{Kh}_i(\psi,\varphi) \in \Gamma, \psi \in \Delta_1, \varphi \in \Delta_2\}.$$
- $\mathrm{S}_i^\Gamma := \left\{\{\langle\psi,\varphi\rangle\} \mid \langle\psi,\varphi\rangle \in \mathsf{Act}_i^\Gamma\right\}$.
- $\mathrm{V}^\Gamma(\Delta) := \{p \in \mathsf{Prop} \mid p \in \Delta\}$. ⊣

Since $\Gamma \in \mathbf{\Phi}$, the structure $\mathcal{M}^\Gamma$ is of the required type, as the following proposition states.

Proposition 5 *The structure* $\mathcal{M}^\Gamma = \langle \mathrm{W}^\Gamma, \mathrm{R}^\Gamma, \{\mathrm{S}_i^\Gamma\}_{i\in\mathsf{Agt}}, \mathrm{V}^\Gamma, \mathsf{Act}^\Gamma\rangle$ *is an* LTS^U.

Proof. It is enough to show that each S_i^Γ defines a partition over a non-empty subset of $2^{(\mathsf{Act}^*)}$. First, $\mathcal{COND}$ implies $\mathsf{Kh}_i(\bot,\bot) \in \Gamma$, so $\langle\bot,\bot\rangle \in \mathsf{Act}_i^\Gamma$ and hence

$\{\langle\bot,\bot\rangle\} \in \mathrm{S}_i^\Gamma$; thus, $\bigcup_{\pi\in \mathrm{S}_i}\pi \neq \varnothing$. Then, S_i indeed defines a partition over $\bigcup_{\pi\in\mathrm{S}_i}\pi$: its elements are mutually disjoint (they are singletons with different elements), collective exhaustiveness is immediate and, finally, $\varnothing \notin \mathrm{S}_i^\Gamma$. ■

Let $\Gamma \in \mathbf{\Phi}$; the following properties of $\mathcal{M}^\Gamma$ will be useful (note that proofs are similar to their counterparts in [60]).

Proposition 6 *For any $\Delta_1, \Delta_2 \in \mathrm{W}^\Gamma$ we have $\Delta_1|_{\mathsf{Kh}} = \Delta_2|_{\mathsf{Kh}}$.*

Proof. Straightforward from the definition of W^Γ. ■

Proposition 7 *Take $\Delta \in \mathrm{W}^\Gamma$. If Δ has a $\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$-successor, then every $\Delta' \in \mathrm{W}^\Gamma$ with $\varphi \in \Delta'$ can be $\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$-reached from Δ.*

Proof. If Δ has a $\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$-successor, then it has a $\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle^i}$-successor for some $i \in \mathsf{Agt}$; thus, $\psi \in \Delta$ and $\mathsf{Kh}_i(\psi,\varphi) \in \Gamma$. Hence, every $\Delta' \in \mathrm{W}^\Gamma$ with $\varphi \in \Delta'$ is such that $(\Delta,\Delta') \in \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle^i}$, and thus such that $(\Delta,\Delta') \in \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$. ■

Proposition 8 *Let φ be an $\mathsf{L}_{\mathsf{Kh}_i}$-formula. If $\varphi \in \Delta$ for every $\Delta \in \mathrm{W}^\Gamma$, then $\mathsf{A}\varphi \in \Delta$ for every $\Delta \in \mathrm{W}^\Gamma$.*

Proof. First, some facts for any Δ in $\mathrm{W}^\Gamma \subseteq \mathbf{\Phi}$. By definition, $\Delta|_{\mathsf{Kh}} \cup \Delta|_{\neg\mathsf{Kh}}$ is a subset of Δ, and therefore it is consistent. Moreover: any maximally consistent extension of $\Delta|_{\mathsf{Kh}} \cup \Delta|_{\neg\mathsf{Kh}}$, say Δ', should satisfy $\Delta|_{\mathsf{Kh}} = \Delta'|_{\mathsf{Kh}}$. For **(⊆)**, note that $\mathsf{Kh}_i(\psi,\varphi) \in \Delta|_{\mathsf{Kh}}$ implies $\mathsf{Kh}_i(\psi,\varphi) \in (\Delta|_{\mathsf{Kh}} \cup \Delta|_{\neg\mathsf{Kh}})$, and thus $\mathsf{Kh}_i(\psi,\varphi) \in \Delta'$, i.e., $\mathsf{Kh}_i(\psi,\varphi) \in \Delta'|_{\mathsf{Kh}}$. For **(⊇)**, use the contrapositive. If $\mathsf{Kh}_i(\psi,\varphi) \notin \Delta|_{\mathsf{Kh}}$ then $\mathsf{Kh}_i(\psi,\varphi) \notin \Delta$, so $\neg\mathsf{Kh}_i(\psi,\varphi) \in \Delta$ (as Δ is an MCS). Thus, $\neg\mathsf{Kh}_i(\psi,\varphi) \in (\Delta|_{\mathsf{Kh}} \cup \Delta|_{\neg\mathsf{Kh}})$ and hence $\neg\mathsf{Kh}_i(\psi,\varphi) \in \Delta'$; therefore, $\mathsf{Kh}_i(\psi,\varphi) \notin \Delta'$ (as Δ is consistent) and thus $\mathsf{Kh}_i(\psi,\varphi) \notin \Delta'|_{\mathsf{Kh}}$.

For the proof of the proposition, suppose $\varphi \in \Delta$ for every $\Delta \in \mathrm{W}^\Gamma$. Take any $\Delta \in \mathrm{W}^\Gamma$, and note how $\Delta|_{\mathsf{Kh}} = \Gamma|_{\mathsf{Kh}}$. Then, the set $\Delta|_{\mathsf{Kh}} \cup \Delta|_{\neg\mathsf{Kh}} \cup \{\neg\varphi\}$ is inconsistent. Otherwise it could be extended into an MCS $\Delta' \in \mathbf{\Phi}$. By the result in the previous paragraph, this would imply $\Delta'|_{\mathsf{Kh}} = \Delta|_{\mathsf{Kh}}$, so $\Delta'|_{\mathsf{Kh}} = \Gamma|_{\mathsf{Kh}}$ and therefore $\Delta' \in \mathrm{W}^\Gamma$. But then, by the assumption, $\varphi \in \Delta'$, and by construction, $\neg\varphi \in \Delta'$. This would make Δ' inconsistent, a contradiction.

Thus, given that $\Delta|_{\mathsf{Kh}} \cup \Delta|_{\neg\mathsf{Kh}} \cup \{\neg\varphi\}$ is inconsistent, there should be sets $\{\mathsf{Kh}_{b_1}(\psi_1,\varphi_1),\dots,\mathsf{Kh}_{b_n}(\psi_n,\varphi_n)\} \subseteq \Delta|_{\mathsf{Kh}}$ and $\{\neg\mathsf{Kh}_{b'_1}(\psi'_1,\varphi'_1),\dots,\neg\mathsf{Kh}_{b'_m}(\psi'_m,\varphi'_m)\} \subseteq \Delta|_{\neg\mathsf{Kh}}$ such that

$$\vdash \left(\bigwedge_{k=1}^{n} \mathsf{Kh}_{b_k}(\psi_k,\varphi_k) \land \bigwedge_{k=1}^{m} \neg\mathsf{Kh}_{b'_k}(\psi'_k,\varphi'_k)\right) \to \varphi.$$

Hence, by $\mathcal{NECA}$,

$$\vdash \mathsf{A}\left(\left(\bigwedge_{k=1}^{n} \mathsf{Kh}_{b_k}(\psi_k,\varphi_k) \land \bigwedge_{k=1}^{m} \neg\mathsf{Kh}_{b'_k}(\psi'_k,\varphi'_k)\right) \to \varphi\right)$$

and then, by $\mathcal{DISTA}$ and $\mathcal{MP}$,

$$\vdash \mathsf{A}\left(\bigwedge_{k=1}^{n} \mathsf{Kh}_{b_k}(\psi_k,\varphi_k) \land \bigwedge_{k=1}^{m} \neg\mathsf{Kh}_{b'_k}(\psi'_k,\varphi'_k)\right) \to \mathsf{A}\varphi.$$

Now, $\mathsf{Kh}_{b_k}(\psi_k, \varphi_k) \in \Delta|_{\mathsf{Kh}}$ implies (*4KhA* and *MP*) that $\mathsf{AKh}_{b_k}(\psi_k, \varphi_k) \in \Delta$ (for each $k \in [1 \,..\, n]$). Similarly, $\neg\mathsf{Kh}_{b'_k}(\psi'_k, \varphi'_k) \in \Delta|_{\mathsf{Kh}}$ implies (*5KhA* and *MP*) that $\mathsf{A}\neg\mathsf{Kh}_{b'_k}(\psi'_k, \varphi'_k) \in \Delta$ (for each $k \in [1 \,..\, m]$). Thus,

$$\bigwedge_{k=1}^{n} \mathsf{AKh}_{b_k}(\psi_k, \varphi_k) \in \Delta \qquad \text{and} \qquad \bigwedge_{k=1}^{m} \mathsf{A}\neg\mathsf{Kh}_{b'_k}(\psi'_k, \varphi'_k) \in \Delta$$

and hence

$$\bigwedge_{k=1}^{n} \mathsf{AKh}_{b_k}(\psi_k, \varphi_k) \wedge \bigwedge_{k=1}^{m} \mathsf{A}\neg\mathsf{Kh}_{b'_k}(\psi'_k, \varphi'_k) \in \Delta, \text{ so } \mathsf{A}\left(\bigwedge_{k=1}^{n} \mathsf{Kh}_{b_k}(\psi_k, \varphi_k) \wedge \bigwedge_{k=1}^{m} \neg\mathsf{Kh}_{b'_k}(\psi'_k, \varphi'_k)\right) \in \Delta$$

and therefore $\mathsf{A}\varphi \in \Delta$. ∎

Proposition 9 *Take ψ, ψ', φ' in $\mathsf{L}_{\mathsf{Kh}_i}$. Suppose that every $\Delta \in W^\Gamma$ with $\psi \in \Delta$ has a $R^\Gamma_{\langle\psi',\varphi'\rangle}$-successor. Then, $\mathsf{A}(\psi \to \psi') \in \Delta$ for all $\Delta \in W^\Gamma$.*

Proof. Take any $\Delta \in W^\Gamma$. On the one hand, if $\psi \in \Delta$ then, by the supposition, $(\Delta, \Delta') \in R^\Gamma_{\langle\psi',\varphi'\rangle}$ for some Δ'. Hence, from $R^\Gamma_{\langle\psi',\varphi'\rangle}$'s definition, $\psi' \in \Delta$ and thus (maximal consistency) $\psi \to \psi' \in \Delta$. On the other hand, if $\psi \notin \Delta$ then $\neg\psi \in \Delta$ (again, maximal consistency) and thus $\psi \to \psi' \in \Delta$. Thus, $\psi \to \psi' \in \Delta$ for every $\Delta \in W^\Gamma$; then, by Proposition 8, $\mathsf{A}(\psi \to \psi') \in \Delta$ for every $\Delta \in W^\Gamma$. ∎

With these properties at hand, we can prove the truth lemma for $\mathcal{M}^\Gamma$.

Lemma 1 (Truth lemma for $\mathcal{M}^\Gamma$) *Given $\Gamma \in \mathbf{\Phi}$, take the canonical model $\mathcal{M}^\Gamma = \langle W^\Gamma, R^\Gamma, \{S^\Gamma_i\}_{i \in \mathsf{Agt}}, V^\Gamma, \mathsf{Act}^\Gamma\rangle$. Then, for every $\Theta \in W^\Gamma$ and every $\varphi \in \mathsf{L}_{\mathsf{Kh}_i}$,*

$$\mathcal{M}^\Gamma, \Theta \models \varphi \qquad \textit{if and only if} \qquad \varphi \in \Theta.$$

Proof. The proof is by induction on φ. The atom and Boolean cases as usual, so we focus on the *knowing how* case.

Case $\mathsf{Kh}_i(\psi, \varphi)$. (⇒) Suppose $\mathcal{M}^\Gamma, \Theta \models \mathsf{Kh}_i(\psi, \varphi)$, and consider two cases.

- $[\![\psi]\!]^{\mathcal{M}^\Gamma} = \varnothing$. Then, each $\Delta \in W^\Gamma$ is such that $\Delta \notin [\![\psi]\!]^{\mathcal{M}^\Gamma}$, which implies $\psi \notin \Delta$ (by IH) and thus $\neg\psi \in \Delta$ (by maximal consistency). Hence, by Proposition 8, $\mathsf{A}\neg\psi \in \Delta$ for every $\Delta \in W^\Gamma$. In particular, $\mathsf{A}\neg\psi \in \Theta$ and thus, by *SCOND* and *MP*, $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$.
- $[\![\psi]\!]^{\mathcal{M}^\Gamma} \neq \varnothing$. From $\mathcal{M}^\Gamma, \Theta \models \mathsf{Kh}_i(\psi, \varphi)$, there is $\{\langle\psi', \varphi'\rangle\} \in S^\Gamma_i$ such that

  (Kh-*1*) $[\![\psi]\!]^{\mathcal{M}^\Gamma} \subseteq \mathrm{SE}(\{\langle\psi', \varphi'\rangle\})$ and **(Kh-*2*)** $R^\Gamma_{\{\langle\psi',\varphi'\rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma}) \subseteq [\![\varphi]\!]^{\mathcal{M}^\Gamma}$.

  In other words, there is $\langle\psi', \varphi'\rangle \in \mathsf{Act}^\Gamma_a$ such that

  (Kh-*1*) for all $\Delta \in W^\Gamma$, if $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$ then $\Delta \in \mathrm{SE}(\{\langle\psi', \varphi'\rangle\})$, so $\Delta \in \mathrm{SE}(\langle\psi', \varphi'\rangle)$ and therefore Δ has a $R^\Gamma_{\langle\psi',\varphi'\rangle}$-successor.

  (Kh-*2*) for all $\Delta' \in W^\Gamma$, if $\Delta' \in R^\Gamma_{\{\langle\psi',\varphi'\rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma})$ then $\Delta' \in [\![\varphi]\!]^{\mathcal{M}^\Gamma}$.

  This case requires three pieces.

  (1) Take any $\Delta \in W^\Gamma$ with $\psi \in \Delta$. Then, by IH, $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$ and thus, by Item **(Kh-*1*)**, Δ has a $R^\Gamma_{\langle\psi',\varphi'\rangle}$-successor. Thus, every $\Delta \in W^\Gamma$ with $\psi \in \Delta$ has such successor; then (Proposition 9), it follows that $\mathsf{A}(\psi \to \psi') \in \Delta$ for every $\Delta \in W^\Gamma$. In particular, $\mathsf{A}(\psi \to \psi') \in \Theta$.

(2) From $\langle \psi', \varphi' \rangle \in \mathsf{Act}_i^\Gamma$ it follows that $\mathsf{Kh}_i(\psi', \varphi') \in \Gamma$. But $\Theta \in W^\Gamma$, so $\Theta|_{\mathsf{Kh}} = \Gamma|_{\mathsf{Kh}}$ (by definition of W^Γ). Hence, $\mathsf{Kh}_i(\psi', \varphi') \in \Theta$.

(3) Since $[\![\psi]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$, there is $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$. By Item **(Kh-*1*)**, Δ should have at least one $R^\Gamma_{\langle \psi', \varphi' \rangle}$-successor. Then, by Proposition 7, every $\Delta' \in W^\Gamma$ satisfying $\varphi' \in \Delta'$ can be $R^\Gamma_{\langle \psi', \varphi' \rangle}$-reached from Δ; in other words, every $\Delta' \in W^\Gamma$ satisfying $\varphi' \in \Delta'$ is in $R^\Gamma_{\langle \psi', \varphi' \rangle}(\Delta)$. But $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$, so every $\Delta' \in W^\Gamma$ satisfying $\varphi' \in \Delta'$ is in $R^\Gamma_{\langle \psi', \varphi' \rangle}([\![\psi]\!]^{\mathcal{M}^\Gamma})$. Then, by Item **(Kh-*2*)**, every $\Delta' \in W^\Gamma$ satisfying $\varphi' \in \Delta'$ is in $[\![\varphi]\!]^{\mathcal{M}^\Gamma}$. By IH on the latter part, every $\Delta' \in W^\Gamma$ satisfying $\varphi' \in \Delta'$ is such that $\varphi \in \Delta'$. Thus, $\varphi' \to \varphi \in \Delta'$ for every $\Delta' \in W^\Gamma$, and hence (Proposition 8) $\mathsf{A}(\varphi' \to \varphi) \in \Delta'$ for every $\Delta' \in W^\Gamma$. In particular, $\mathsf{A}(\varphi' \to \varphi) \in \Theta$.

Thus, $\{\mathsf{A}(\psi \to \psi'), \mathsf{Kh}_i(\psi', \varphi'), \mathsf{A}(\varphi' \to \varphi)\} \subset \Theta$. Therefore, by $\mathcal{KhA}$ and $\mathcal{MP}$, $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$.

(⇐) Suppose $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$. Thus (Proposition 6), $\mathsf{Kh}_i(\psi, \varphi) \in \Gamma$, so $\langle \psi, \varphi \rangle \in \mathsf{Act}_i^\Gamma$ and therefore $\{\langle \psi, \varphi \rangle\} \in S_i^\Gamma$. The rest of the proof is split into two cases.

- Suppose there is no $\Delta_\psi \in W^\Gamma$ with $\psi \in \Delta$. Then, by IH, there is no $\Delta_\psi \in W^\Gamma$ with $\Delta_\psi \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$, that is, $[\![\neg\psi]\!]^{\mathcal{M}^\Gamma} = D_{W^\Gamma}$. Since $\mathcal{M}^\Gamma$ is an LTS^U (Proposition 5), the latter yields $(\mathcal{M}^\Gamma, \Delta) \models \mathsf{Kh}_i(\psi, \chi)$ for any $i \in \mathsf{Agt}$, $\chi \in \mathsf{L}_{\mathsf{Kh}_i}$ and $\Delta \in W^\Gamma$ (cf. Proposition 2); hence, $(\mathcal{M}^\Gamma, \Theta) \models \mathsf{Kh}_i(\psi, \varphi)$.
- Suppose there is $\Delta_\psi \in W^\Gamma$ with $\psi \in \Delta_\psi$. It will be shown that the set of plans $\{\langle \psi, \varphi \rangle\} \in S_i^\Gamma$ satisfies the requirements.

  (Kh-*1*) Take any $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$. By IH, $\psi \in \Delta$. Moreover, from $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$ and Proposition 6 it follows that $\mathsf{Kh}_i(\psi, \varphi) \in \Delta$. Then, from $R^\Gamma_{\langle \psi, \varphi \rangle^i}$'s definition, every $\Delta' \in W^\Gamma$ with $\varphi \in \Delta'$ is such that $(\Delta, \Delta') \in R^\Gamma_{\langle \psi, \varphi \rangle^i}$, and therefore such that $(\Delta, \Delta') \in R^\Gamma_{\langle \psi, \varphi \rangle}$. Now note how, since there is $\Delta_\psi \in W^\Gamma$ with $\psi \in \Delta_\psi$, there should be $\Delta_\varphi \in W^\Gamma$ with $\varphi \in \Delta_\varphi$. Suppose otherwise, i.e., suppose there is no $\Delta'' \in W^\Gamma$ with $\varphi \in \Delta''$. Then, $\neg\varphi \in \Delta''$ for every $\Delta'' \in W^\Gamma$, and hence (Proposition 8) $\mathsf{A}\neg\varphi \in \Delta''$ for every $\Delta'' \in W^\Gamma$. In particular, $\mathsf{A}\neg\varphi \in \Delta_\psi$. Moreover, from $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$ and Proposition 6 it follows that $\mathsf{Kh}_i(\psi, \varphi) \in \Delta_\psi$. Then, $\mathcal{KhE}$ (written as $\mathsf{Kh}_i(\psi, \varphi) \to (\mathsf{A}\neg\varphi \to \mathsf{A}\neg\psi)$) and $\mathcal{MP}$ yield $\mathsf{A}\neg\psi \in \Delta_\psi$, and thus (axiom $\mathcal{TA}$) $\neg\psi \in \Delta_\psi$. Hence, $\{\psi, \neg\psi\} \subset \Delta_\psi$, contradicting Δ_ψ's consistency.
  Let us continue with the proof of the lemma. The existence of $\Delta_\varphi \in W^\Gamma$ with $\varphi \in \Delta_\varphi$ implies that $(\Delta, \Delta_\varphi) \in R^\Gamma_{\langle \psi, \varphi \rangle}$ and thus, since $\langle \psi, \varphi \rangle$ is a basic action, $\Delta \in SE(\langle \psi, \varphi \rangle)$, and so $\Delta \in SE(\{\langle \psi, \varphi \rangle\})$. Since Δ is an arbitrary state in $[\![\psi]\!]^{\mathcal{M}^\Gamma}$, the required $[\![\psi]\!]^{\mathcal{M}^\Gamma} \subseteq SE(\{\langle \psi, \varphi \rangle\})$ follows.

  (Kh-*2*) Take any $\Delta' \in R^\Gamma_{\{\langle \psi, \varphi \rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma})$. Then, there is $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$ such that $(\Delta, \Delta') \in R^\Gamma_{\langle \psi, \varphi \rangle}$. By definition of R^Γ, it follows that $\varphi \in \Delta'$ so, by IH, $\Delta' \in [\![\varphi]\!]^{\mathcal{M}^\Gamma}$. Since Δ' is an arbitrary state in $R^\Gamma_{\{\langle \psi, \varphi \rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma})$, the required $R^\Gamma_{\{\langle \psi, \varphi \rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma}) \subseteq [\![\varphi]\!]^{\mathcal{M}^\Gamma}$ follows. ■

Finally, we present the intended result.

Theorem 4 *The axiom system $\mathcal{L}^{\mathrm{LTS}^U}_{\mathsf{Kh}_i}$ (Table 2) is sound and strongly complete for $\mathsf{L}_{\mathsf{Kh}_i}$ w.r.t. the class of all* LTS^U*s.*

Proof. For soundness, it is enough to show that the system's axioms are valid and that its rules preserve validity (which, as mentioned before, is straightforward). For completeness, take any $\mathcal{L}^{\mathrm{LTS}^U}_{\mathsf{Kh}_i}$-consistent set of formulas $\Gamma' \subseteq \mathsf{L}_{\mathsf{Kh}_i}$. Since $\mathsf{L}_{\mathsf{Kh}_i}$ is enumerable, Γ' can be extended into a maximally $\mathcal{L}^{\mathrm{LTS}^U}_{\mathsf{Kh}_i}$-consistent set $\Gamma \supseteq \Gamma'$ by a standard Lindenbaum's construction (see, e.g., [10, Lemma 4.17]). By Lemma 1, Γ' is satisfiable in $\mathcal{M}^\Gamma$ at Γ. The fact that $\mathcal{M}^\Gamma$ is an LTS^U (Proposition 5) completes the proof. ∎

One detail in the construction of the canonical model might be surprising: each set of indistinguishable plans for a given agent is a singleton set. Hence, the logic is also complete with respect to this particular class of models. On the other hand, LTS^U*s* are a more general and accurate representation from a conceptual point of view. We could, for instance, extend the language so that this representation is also reflected by the logic (i.e., the language can explicitly refer to plans), or define public announcements-like modalities to refine the indistinguishability relation of each agent (see, e.g., [5]).

7 Recapturing the Original Logic of Knowing How

This section compares the original logic from [58, 59, 60] with the one introduced in this paper. More precisely, first it will be shown that our logic is weaker than the original one. Then, we will explore two different classes of LTS^{U}s under which we can capture the exact semantics of [58, 59, 60]. As a consequence, the axiom system in Table 1 is sound and complete for these two classes of models. This shows that our framework generalizes the one based on LTSs. For the comparison to be meaningful, we will restrict the LTS^U setting to its single-agent case: a single modality Kh and no subindexes for P_i and S_i.

The provided axiom system can be used to compare the notion of *knowing how* under LTSs with that under LTS^{U}s. Here is a first observation.

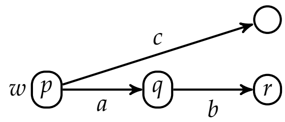
Proposition 10 *$\mathcal{KhE}$ and $\mathcal{KhA}$ are theorems of $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$.*

Proof. $\mathcal{KhE}$ can be rewritten as $(\mathsf{Kh}(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \rightarrow \mathsf{A}\neg\psi$, which is an instance of $\mathcal{COMPKh}$ in $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$ (just unfold A). For $\mathcal{KhA}$, use $\mathcal{EMP}$ and then $\mathcal{COMPKh}$ [60, Proposition 2]. ∎

Hence, the *knowing how* operator under LTSs is at least as strong as its LTS^U-based counterpart: every formula valid under LTS^{U}s is also valid under LTSs. The following fact shows that the converse is not the case.

Proposition 11 *Within* LTS^U*, axioms $\mathcal{EMP}$ and $\mathcal{COMPKh}$ are not valid.*

Proof. Consider the LTS^U shown below, with the collection of sets of plans for the agent (i.e., the set S) depicted on the right. Recall that Kh acts globally.



$$S = \left\{ \begin{array}{c} \{a\}, \{b\} \\ \{ab, c\} \end{array} \right\}$$

With respect to $\mathcal{EMP}$, notice that $\mathsf{A}(p \to p)$ holds; yet, $\mathsf{Kh}(p, p)$ fails since there is no $\pi \in S$ leading from p-states to p-states. More generally, $\mathcal{EMP}$ is valid over LTSs because the empty plan ϵ, strongly executable everywhere, is always available. However, in an LTS^U, the plan ϵ might not be available to the agent (i.e., $\epsilon \notin P$), and even if it is, it might be indistinguishable from other plans with different behaviour.

With respect to $\mathcal{COMPKh}$, notice that $\mathsf{Kh}(p, q)$ and $\mathsf{Kh}(q, r)$ hold, witness $\{a\}$ and $\{b\}$, respectively. However, there is no $\pi \in S$ containing only plans that, when started on p-states, lead only to r-states. Thus, $\mathsf{Kh}(p, r)$ fails. More generally, $\mathcal{COMPKh}$ is valid over LTS because the sequential composition of the plans that make true the conjuncts in the antecedent is a witness that makes true the consequent. However, in an LTS^U, this composition might be unavailable or indistinguishable from other plans. ■

From these two observations it follows that Kh under LTS^Us is strictly weaker than Kh under LTSs: adding uncertainty about plans changes the logic.

7.1 A very simple class of LTS^Us

Still, the uncertainty-based framework is general enough to capture the LTS semantics. Given the discussion in Proposition 11, there is an obvious class of LTS^Us in which $\mathcal{EMP}$ and $\mathcal{COMPKh}$ are valid: the class of LTS^Us with no uncertainty, in which the agent has every plan available and can distinguish between any two of them. Below, we define formally this class.

Definition 7.1 Define the class of models:

$$\mathbf{M_{NU}} := \{\mathcal{M} \mid \mathcal{M} = \langle W, R, S, V, \mathsf{Act}\rangle \text{ is an } \text{LTS}^U \text{ and } S = \{\{\sigma\} \mid \sigma \in \mathsf{Act}^*\}\}. \quad \dashv$$

Indeed, for models in $\mathbf{M_{NU}}$, the plan ϵ is available and distinguishable from other plans (witnessing $\mathcal{EMP}$) and from $\{\sigma_1\} \in S$ and $\{\sigma_2\} \in S$ it follows that $\{\sigma_1\sigma_2\} \in S$ (witnessing $\mathcal{COMPKh}$). Thus, as the following proposition states, an agent in LTS is exactly an agent in LTS^U that can use every plan and has no uncertainty and full awareness about them. This class is enough to show how the uncertainty-based framework can capture the original one.

Proposition 12 *The following properties hold.*

(1) *Given a model* $\mathcal{M} = \langle W, R, S, V, \mathsf{Act}\rangle$ *in* $\mathbf{M_{NU}}$, *the* LTS $\mathcal{S}_{\mathcal{M}} = \langle W, R, V, \mathsf{Act}\rangle$ *is such that* $[\![\varphi]\!]^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{S}_{\mathcal{M}}}$ *for every* $\varphi \in \mathsf{L_{Kh}}$.

(2) *Given an* LTS $\mathcal{S} = \langle W, R, V, \mathsf{Act}\rangle$, *the model* $\mathcal{M}_{\mathcal{S}} = \langle W, R, S, V, \mathsf{Act}\rangle$ *with* $S = \{\{\sigma\} \mid \sigma \in \mathsf{Act}^*\}$, *is in* $\mathbf{M_{NU}}$ *and is such that* $[\![\varphi]\!]^{\mathcal{S}} = [\![\varphi]\!]^{\mathcal{M}_{\mathcal{S}}}$ *for every* $\varphi \in \mathsf{L_{Kh}}$. ◁

This correspondence, showing that every LTS has a point-wise equivalent model in $\mathbf{M_{NU}}$ and vice-versa, gives us a direct completeness result.

Theorem 5 *The axiom system $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$ (Table 1) is sound and strongly complete for L_{Kh} w.r.t. the class $\mathbf{M_{NU}}$.*

Proof. For soundness, we look at both blocks in Table 1. For the first, Theorem 4 shows that those axioms and rules are sound for all LTS^{U}, and thus in particular sound for those in the class $\mathbf{M_{NU}}$. For the second, Item ***(1)*** of Proposition 12 shows that every model in $\mathbf{M_{NU}}$ is point-wise L_{Kh}-equivalent to an LTS, thus (Theorem 1) making sound such axioms.

To prove that $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$ is strongly complete over the class $\mathbf{M_{NU}}$, we need to show that, given $\Gamma \cup \{\varphi\}$ a set of formulas in L_{Kh}, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$. Let Γ be a consistent set of formulas. As in [60, Lemma 1], Γ can be extended to an MCS Γ', and as a consequence, there exists an LTS $\mathcal{S}^{\Gamma'}$ such that $\mathcal{S}^{\Gamma'}, \Gamma' \models \Gamma$ (notice that states in the canonical model are MCS). Then, by Item ***(2)*** of Proposition 12, we can obtain an LTS^{U} $\mathcal{M}_{\mathcal{S}^{\Gamma'}}$, such that $\mathcal{M}_{\mathcal{S}^{\Gamma'}}, \Gamma' \models \Gamma$. Moreover, from Item ***(2)*** of Proposition 12 we also know that $\mathcal{M}_{\mathcal{S}^{\Gamma'}}$ is in $\mathbf{M_{NU}}$. ■

7.2 Active and compositional LTS^{U}s

We presented above a very simple class of models that enables us to establish a direct relation between both semantics. However, the result is somewhat trivial: LTS^{U}s generalize LTSs by adding uncertainty among plans, and the class $\mathbf{M_{NU}}$ contains those LTS^{U}s in which the agent does not have uncertainty. The rest of this section will discuss a larger and more general class (with weaker constraints) for which the same correspondence holds.

Let us start by introducing some preliminary definitions.

Definition 7.2 Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{S}, \mathrm{V}, \mathsf{Act}\rangle$ be an LTS^{U}. The *composition* of $\pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$ is the set of plans $\pi_1\pi_2 \in 2^{\mathsf{Act}^*}$ given by

$$\pi_1\pi_2 := \{\sigma_1\sigma_2 \in \mathsf{Act}^* \mid \sigma_1 \in \pi_1 \text{ and } \sigma_2 \in \pi_2\}.$$

Now, here there are the crucial properties we will require of LTS^{U}, to establish the intended correspondence with LTSs.

Definition 7.3 We say that an LTS^{U} $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{S}, \mathrm{V}, \mathsf{Act}\rangle$ is:

- *active* if and only if there exists $\pi \in \mathrm{S}$ such that $\mathrm{SE}(\pi) = \mathrm{W}$ and, for all $u, v \in \mathrm{W}$, $v \in \mathrm{R}_{\pi}(u)$ implies $\mathcal{M}, u \leftrightarrows \mathcal{M}, v$.
- *compositional* if and only if for all $\pi_1, \pi_2 \in \mathrm{S}$ there exists $\pi \in \mathrm{S}$ such that:

  (1) $\mathrm{R}_{\pi_1\pi_2} = \mathrm{R}_{\pi}$, and

  (2) $\mathrm{SE}(\pi_1\pi_2) \subseteq \mathrm{SE}(\pi)$.

We define the class $\mathbf{M_{AC}} := \{\mathcal{M} \mid \mathcal{M} \text{ is active and compositional}\}$. ⊣

While activeness ensures that there is a set of plans doing what the empty plan ϵ does in an LTS, compositionality ensures that S is closed under a suitable notion of composition of sets of plans.

The next lemma establishes that the requirements for compositionality generalize to an arbitrary number of sets of plans.

Lemma 2 *Let* $\mathcal{M} = \langle W, R, S, V, \mathsf{Act}\rangle$ *be a compositional* LTS^U*, and* $\pi_1, \ldots, \pi_k \in S$ *(with* $k \geq 2$*). Then, there is* $\pi \in S$ *such that:*

(1) $R_{\pi_1\cdots\pi_k} = R_\pi$*, and*

(2) $SE(\pi_1 \cdots \pi_k) \subseteq SE(\pi)$.

Proof. We prove the existence of π by induction on $k \geq 2$; then we will show that this witness does the work. The base case $k = 2$ follows from the definition, so take sets of plans in $\pi_1, \ldots, \pi_k, \pi_{k+1} \in S$. By inductive hypothesis, there is a $\pi' \in S$ such that:

- $R_{\pi_2\cdots\pi_{k+1}} = R_{\pi'}$,
- $SE(\pi_2 \cdots \pi_{k+1}) \subseteq SE(\pi')$.

Using the definition of compositionality there is $\pi \in S$ s.t.

- $R_{\pi_1\pi'} = R_\pi$,
- $SE(\pi_1\pi') \subseteq SE(\pi)$.

We will prove that π is the witness we are looking for.

For Item ***(1)***, the following chain of equalities hold: $R_{\pi_1\cdots\pi_{k+1}} = R_{\pi_1} \circ R_{\pi_2\cdots\pi_{k+1}} = R_{\pi_1} \circ R_{\pi'} = R_{\pi_1\pi'} = R_\pi$.

For Item ***(2)***, take $w \in SE(\pi_1 \cdots \pi_{k+1})$. Then $w \in SE(\pi_1)$ and for all $v \in R_{\pi_1}(w)$ we have that $v \in SE(\pi_2 \cdots \pi_{k+1})$. Thus, $w \in SE(\pi_1)$ and for all $v \in R_{\pi_1}(w)$ we have that $v \in SE(\pi')$. Hence $w \in SE(\pi_1\pi') \subseteq SE(\pi)$. ■

With these tools at hand, we will show that for every LTS there is an $\mathsf{L_{Kh}}$-equivalent LTS^U in $\mathbf{M_{AC}}$, and vice-versa. First, we present the mapping from LTS^Us to LTSs.

Proposition 13 *Let* $\mathcal{M} = \langle W, R, S, V, \mathsf{Act}\rangle$ *be an* LTS^U *in* $\mathbf{M_{AC}}$*. Let us take* $\mathsf{Act}' := \{a_\pi \mid \pi \in S\}$*, and then define the LTS* $\mathcal{S}_\mathcal{M} = \langle W, R', V, \mathsf{Act}'\rangle$ *by taking* $R'_{a_\pi} := \{(w, v) \in R_\pi \mid w \in SE(\pi)\}$ *(so basic actions in* $\mathcal{S}_\mathcal{M}$ *correspond to sets of SE plans in* $\mathcal{M}$*). Then,* $[\![\varphi]\!]^\mathcal{M} = [\![\varphi]\!]^{\mathcal{S}_\mathcal{M}}$ *for every* $\varphi \in \mathsf{L_{Kh}}$.

Proof. It is clear that $\mathcal{S}_\mathcal{M}$ is an LTS. The rest of the proof is by structural induction on the formulas in $\mathsf{L_{Kh}}$. The cases for the Boolean fragment are straightforward. We will discuss the case for formulas of the shape $\mathsf{Kh}(\psi, \varphi)$.

In doing so, the following property will be useful: for every $\pi \in S$, we have $SE(\pi) = SE(a_\pi)$ and for all $u \in SE(\pi)$, $R_\pi(u) = R'_{a_\pi}(u)$. Indeed, **(⊆)** if $u \in SE(\pi)$ then there is $v \in W$ such that $(u, v) \in R_\pi$, so $(u, v) \in R'_{a_\pi}$ and therefore, being a_π a basic action, $u \in SE(a_\pi)$. Moreover, **(⊇)** if $u \in SE(a_\pi)$ then there is $v \in W$ such that $(u, v) \in R'_{a_\pi}$, so $u \in SE(\pi)$. The second part of the property is direct.

This can be generalized as $SE(\pi_1 \cdots \pi_k) = SE(a_{\pi_1} \cdots a_{\pi_k})$ and for all $u \in SE(\pi_1 \cdots \pi_k)$, $R_{\pi_1\cdots\pi_k}(u) = R'_{a_{\pi_1}\cdots a_{\pi_k}}(u)$ with $k \geq 1$. The base case was proved above. Let $u \in SE(\pi_1 \cdots \pi_{k+1})$, using the definition of SE, $u \in SE(\pi_1 \cdots \pi_k)$ and for all $v \in R_{\pi_1\cdots\pi_k}(u)$, then $v \in SE(\pi_{k+1})$. Using the IH and the base case, $u \in SE(a_{\pi_1} \cdots a_{\pi_k})$ and for all $v \in R'_{a_{\pi_1}\cdots a_{\pi_k}}(u)$, then $v \in SE(a_{\pi_{k+1}})$. Which is equivalent to $u \in SE(a_{\pi_1} \cdots a_{\pi_{k+1}})$. For the second part, $(u, z) \in R_{\pi_1\cdots\pi_{k+1}}$ iff there is $v \in W$ s.t. $(u, v) \in R_{\pi_1\cdots\pi_k}$ and $(v, z) \in R_{\pi_{k+1}}$. Since $u \in SE(\pi_1 \cdots \pi_{k+1})$, then $u \in SE(\pi_1 \cdots \pi_k)$ and for all $v \in R_{\pi_1\cdots\pi_k}(u)$, then $v \in SE(\pi_{k+1})$. Hence, there is $v \in W$ s.t. $(u, v) \in R'_{a_{\pi_1}\cdots a_{\pi_k}}$ and $(v, z) \in R'_{a_{\pi_{k+1}}}$. Thus, $(u, z) \in R'_{a_{\pi_1}\cdots a_{\pi_{k+1}}}$.

(⊆) Suppose $w \in [\![\mathsf{Kh}(\psi, \varphi)]\!]^\mathcal{M}$; then there is $\pi \in S$ satisfying both

(Kh-1) $[\![\psi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and ***(Kh-2)*** $\mathrm{R}_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$.

We will prove $w \in [\![\mathsf{Kh}(\psi,\varphi)]\!]^{\mathcal{S}_\mathcal{M}}$ using $a_\pi \in \mathsf{Act}'$ as our witness. First, for showing that a_π has the right properties, suppose $v \in [\![\psi]\!]^{\mathcal{S}_\mathcal{M}}$. Then $v \in [\![\psi]\!]^{\mathcal{M}}$ (by IH), so $v \in \mathrm{SE}(\pi)$ (by Item **(Kh-*1*)**) and hence $v \in \mathrm{SE}(a_\pi)$ (property discussed above). Therefore, $[\![\psi]\!]^{\mathcal{S}_\mathcal{M}} \subseteq \mathrm{SE}(a_\pi)$. Second, for showing that a_π does the required work, suppose $u \in \mathrm{R}'_{a_\pi}([\![\psi]\!]^{\mathcal{S}_\mathcal{M}})$. Then, $u \in \mathrm{R}'_{a_\pi}([\![\psi]\!]^{\mathcal{M}})$ (by IH), hence $u \in \mathrm{R}_\pi([\![\psi]\!]^{\mathcal{M}})$ (by definition of R′) so $u \in [\![\varphi]\!]^{\mathcal{M}}$ (by Item **(Kh-*2*)**), and then $u \in [\![\varphi]\!]^{\mathcal{S}_\mathcal{M}}$ (by IH). Thus, $\mathrm{R}'_{a_\pi}([\![\psi]\!]^{\mathcal{S}_\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{S}_\mathcal{M}}$. From the two pieces, it follows that $w \in [\![\mathsf{Kh}(\psi,\varphi)]\!]^{\mathcal{S}_\mathcal{M}}$.

(⊇) Suppose $w \in [\![\mathsf{Kh}(\psi,\varphi)]\!]^{\mathcal{S}_\mathcal{M}}$; then there is $\sigma \in (\mathsf{Act}')^*$ satisfying both

(Kh-1) $[\![\psi]\!]^{\mathcal{S}_\mathcal{M}} \subseteq \mathrm{SE}(\sigma)$ and ***(Kh-2)*** $\mathrm{R}'_\sigma([\![\psi]\!]^{\mathcal{S}_\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{S}_\mathcal{M}}$.

There are two main cases. First, assume $\sigma = \epsilon$. Since $\mathcal{M}$ is active, there is $\pi \in \mathrm{S}$ s.t. $\mathrm{SE}(\pi) = \mathrm{W}$ and for all $u, v \in \mathrm{W}$, $v \in \mathrm{R}_\pi(u)$ implies $\mathcal{M}, u \leftrightarrows \mathcal{M}, v$. It is not hard to show that this is the witness we need.

Second, assume $\sigma \neq \epsilon$, i.e., $\sigma = a_{\pi_1} \cdots a_{\pi_k}$ with $a_{\pi_i} \in \mathsf{Act}'$ (so $\pi_i \in \mathrm{S}$). Then, we contemplate two scenarios. If $[\![\psi]\!]^{\mathcal{S}_\mathcal{M}} = \varnothing$ then, by IH, $[\![\psi]\!]^{\mathcal{M}} = \varnothing$; thus, any $\pi \in \mathrm{S}$ works as a witness. Otherwise, $[\![\psi]\!]^{\mathcal{S}_\mathcal{M}} \neq \varnothing$ and then, since $\mathcal{M}$ is compositional, by Lemma 2 there exists $\pi \in \mathrm{S}$ such that $\mathrm{R}_{\pi_1 \cdots \pi_k} = \mathrm{R}_\pi$ and $\mathrm{SE}(\pi_1 \cdots \pi_k) \subseteq \mathrm{SE}(\pi)$.

For the first Kh-clause, take $v \in [\![\psi]\!]^{\mathcal{M}}$. Then, by IH, $v \in [\![\psi]\!]^{\mathcal{S}_\mathcal{M}}$, and by Item **(Kh-*1*)**, $v \in \mathrm{SE}(a_{\pi_1} \cdots a_{\pi_k})$. Using the properties proved at the beginning, $v \in \mathrm{SE}(\pi_1 \cdots \pi_k)$. Thus, $v \in \mathrm{SE}(\pi)$. For the second Kh-clause, take $u, v \in \mathrm{W}$ such that $u \in [\![\psi]\!]^{\mathcal{M}}$ and $(u, v) \in \mathrm{R}_\pi = \mathrm{R}_{\pi_1 \cdots \pi_k}$. By IH, $u \in [\![\psi]\!]^{\mathcal{S}_\mathcal{M}}$. Since $u \in \mathrm{SE}(\pi_1 \cdots \pi_k) = \mathrm{SE}(a_{\pi_1} \cdots a_{\pi_k})$, then $(u, v) \in \mathrm{R}'_{a_{\pi_1} \cdots a_{\pi_k}}$ and by Item **(Kh-*2*)** $v \in [\![\varphi]\!]^{\mathcal{S}_\mathcal{M}}$. Again, by IH $v \in [\![\varphi]\!]^{\mathcal{M}}$. Therefore $\mathrm{R}_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$. From the two pieces, $w \in [\![\mathsf{Kh}(\psi,\varphi)]\!]^{\mathcal{M}}$.

This finishes the proof. ■

Now we will prove the other direction. From an LTS we can obtain an active, compositional and point-wise equivalent LTS^U.

Proposition 14 *Let $\mathcal{S} = \langle \mathrm{W}, \mathrm{R}, \mathrm{V}, \mathsf{Act} \rangle$ be an* LTS. *Let us take $\mathsf{Act}' := \{a_b \mid b \in \mathsf{Act} \cup \{\epsilon\}\}$, and then define the LTS^U $\mathcal{M}_\mathcal{S} = \langle \mathrm{W}, \mathrm{R}', \mathrm{S}', \mathrm{V}, \mathsf{Act}' \rangle$ by taking $\mathrm{R}'_{a_b} := \mathrm{R}_b$, and $\mathrm{S}' := \{\{\alpha\} \mid \alpha \in (\mathsf{Act}')^*\}$. Then,*

(1) $\mathcal{M}_\mathcal{S}$ is an active and compositional LTS^U (i.e., is in $\mathbf{M_{AC}}$);

(2) for every $\varphi \in \mathsf{L}_{\mathsf{Kh}}$, $[\![\varphi]\!]^{\mathcal{S}} = [\![\varphi]\!]^{\mathcal{M}_\mathcal{S}}$.

Proof. First, Item ***(1)***. For showing that $\mathcal{M}_\mathcal{S}$ is an LTS^U, note how $\mathrm{P}' = \bigcup_{\pi \in \mathrm{S}'} \pi$ is non-empty ($\epsilon \in \mathsf{Act}^*$, so $a_\epsilon \in \mathsf{Act}'$ and hence $\{a_\epsilon\} \in \mathrm{S}'$) and, moreover, S′ does not contain the empty set and its elements are pairwise disjoint (the latter two by definition).

Activeness is straightforward, as $\{a_\epsilon\}$ is in S′ and behaves exactly as ϵ. For compositionality, let $\pi_1 = \{a_{b_1} \ldots a_{b_n}\}, \pi_2 = \{a_{c_1} \ldots a_{c_m}\} \in \mathrm{S}'$. Since $\pi_1\pi_2 = \{a_{b_1} \ldots a_{b_n} a_{c_1} \ldots a_{c_m}\} = \pi \in \mathrm{S}'$, the two conditions hold since $\mathrm{R}_{\pi_1\pi_2} = \mathrm{R}_\pi$ and $\mathrm{SE}(\pi_1\pi_2) = \mathrm{SE}(\pi)$.

Something that comes out from $\mathcal{M}_\mathcal{S}$ is that for all $b \in \mathsf{Act}$, $\mathrm{SE}(b) = \mathrm{SE}(a_b)$ and if $u \in \mathrm{SE}(b)$, then $\mathrm{R}_b(u) = \mathrm{R}'_{a_b}(u)$. This can be generalized to $\sigma = b_1 \ldots b_n \in \mathsf{Act}^*$,

$\mathrm{SE}(b_1 \dots b_n) = \mathrm{SE}(a_{b_1} \dots a_{b_n})$ and if $u \in \mathrm{SE}(a_{b_1} \dots a_{b_n})$, then $\mathrm{R}_{b_1 \dots b_n}(u) = \mathrm{R}'_{a_{b_1} \dots a_{b_n}}(u)$. The base case is direct. Let $\sigma = b_1 \dots b_n b_{n+1}$ and $u \in \mathrm{SE}(b_1 \dots b_{n+1})$. Thus, $u \in \mathrm{SE}(b_1 \dots b_n)$ and for all $v \in \mathrm{R}_{b_1 \dots b_n}(u)$, $v \in \mathrm{SE}(b_{n+1})$. By IH, $u \in \mathrm{SE}(a_{b_1} \dots a_{b_n})$ and for all $v \in \mathrm{R}_{a_{b_1} \dots a_{b_n}}(u)$, $v \in \mathrm{SE}(a_{b_{n+1}})$. This is equivalent to $u \in \mathrm{SE}(a_{b_1} \dots a_{b_{n+1}})$. Let $u \in \mathrm{SE}(b_1 \dots b_{n+1})$ and $v \in \mathrm{R}_{b_1 \dots b_{n+1}}(u)$, then $u \in \mathrm{SE}(b_1 \dots b_n)$ and there is $w \in \mathrm{R}_{b_1 \dots b_n}(u)$ s.t. $(w, v) \in \mathrm{R}_{b_{n+1}}$ and $w \in \mathrm{SE}(b_{n+1})$. By IH and the definition of R', $w \in \mathrm{R}'_{a_{b_1} \dots a_{b_n}}(u)$ and $(w, v) \in \mathrm{R}'_{a_{b_{n+1}}}$. This is equivalent to $v \in \mathrm{R}'_{a_{b_1} \dots a_{b_{n+1}}}(u)$

For Item *(2)*, the proof is by structural induction; again, only the case of $\mathsf{Kh}(\psi, \varphi)$ is discussed.

(⊆) Suppose $w \in [\![\mathsf{Kh}(\psi, \varphi)]\!]^{\mathcal{S}}$; then there is $\sigma = b_1 \dots b_n \in \mathsf{Act}^*$ satisfying both

(Kh-1) $[\![\psi]\!]^{\mathcal{S}} \subseteq \mathrm{SE}(\sigma)$ and ***(Kh-2)*** $\mathrm{R}_\sigma([\![\psi]\!]^{\mathcal{S}}) \subseteq [\![\varphi]\!]^{\mathcal{S}}$.

There are two cases. First, if $[\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}} = \varnothing$, then any $\pi \in \mathrm{S}'$ will work. Hence, to obtain $w \in [\![\mathsf{Kh}(\psi, \varphi)]\!]^{\mathcal{M}_{\mathcal{S}}}$ it is enough to have $\mathrm{S}' \neq \varnothing$, which we do as $\{a_\epsilon\} \in \mathrm{S}'$.

Second, if $[\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}} \neq \varnothing$, $\{a_{b_1} \dots a_{b_n}\} \in \mathrm{S}'$ will be shown to be our witness. For the first Kh-clause, if $u \in [\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}}$ then $u \in [\![\psi]\!]^{\mathcal{S}}$ (IH), so $u \in \mathrm{SE}(\sigma)$. Using the properties proven before, $u \in \mathrm{SE}(a_{b_1} \dots a_{b_n})$. For the second Kh-clause, suppose $v \in \mathrm{R}'_{a_{b_1} \dots a_{b_n}}([\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}})$. Then, $v \in \mathrm{R}'_{a_{b_1} \dots a_{b_n}}(u)$ for some $u \in [\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}} \subseteq \mathrm{SE}(a_{b_1} \dots a_{b_n})$. Then, $v \in \mathrm{R}_{b_1 \dots b_n}(u)$ and $u \in [\![\psi]\!]^{\mathcal{S}}$ (IH). Hence, $v \in [\![\varphi]\!]^{\mathcal{S}}$ and $v \in [\![\varphi]\!]^{\mathcal{M}_{\mathcal{S}}}$ (IH). Consequently, $\mathrm{R}'_\pi([\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}}) \subseteq [\![\varphi]\!]^{\mathcal{M}_{\mathcal{S}}}$. From the two clauses, $w \in [\![\mathsf{Kh}(\psi, \varphi)]\!]^{\mathcal{M}_{\mathcal{S}}}$.

(⊇) Suppose $w \in [\![\mathsf{Kh}(\psi, \varphi)]\!]^{\mathcal{M}_{\mathcal{S}}}$. Then there is an element of S' fulfilling the Kh-clauses, which by definition of S' implies there is $\pi_\sigma = \{a_{b_1} \dots a_{b_n}\} \in \mathrm{S}'$ (with $\sigma = b_1 \dots b_n \in \mathsf{Act}^*$) satisfying both

(Kh-1) $[\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}} \subseteq \mathrm{SE}(\pi_\sigma)$ and ***(Kh-2)*** $\mathrm{R}'_{\pi_\sigma}([\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}}) \subseteq [\![\varphi]\!]^{\mathcal{M}_{\mathcal{S}}}$.

It will be shown that σ is our witness. For the first Kh-clause, $u \in [\![\psi]\!]^{\mathcal{S}}$ implies $u \in [\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}}$ (IH), hence $u \in \mathrm{SE}(\pi_\sigma)$ (Item ***(Kh-1)***) and then $u \in \mathrm{SE}(\sigma)$ using the properties shown at the beginning of the proof. Thus, $[\![\psi]\!]^{\mathcal{S}} \subseteq \mathrm{SE}(\sigma)$. For the second Kh-clause, take $u \in \mathrm{R}_\sigma([\![\psi]\!]^{\mathcal{S}})$, so $u \in \mathrm{R}_\sigma([\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}})$ (IH). Then $u \in \mathrm{R}_\sigma(v)$ for some $v \in [\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}}$ By Item ***(Kh-1)***, $v \in \mathrm{SE}(\pi_\sigma)$, thus $u \in \mathrm{R}'_{\pi_\sigma}(v)$. Hence, $u \in \mathrm{R}'_{\pi_\sigma}([\![\psi]\!]^{\mathcal{M}_{\mathcal{S}}})$, so $u \in [\![\varphi]\!]^{\mathcal{M}_{\mathcal{S}}}$ (Item ***(Kh-2)***) and then $u \in [\![\varphi]\!]^{\mathcal{S}}$ (IH). Thus, $\mathrm{R}_\sigma([\![\psi]\!]^{\mathcal{S}}) \subseteq [\![\varphi]\!]^{\mathcal{S}}$. From the two clauses, we get $w \in [\![\mathsf{Kh}(\psi, \varphi)]\!]^{\mathcal{S}}$. ■

From these results, the axiom system for $\mathsf{L_{Kh}}$ over LTS (Table 1) is also sound and complete for $\mathsf{L_{Kh}}$ over *active* and *compositional* LTS^Us.

Theorem 6 *The axiom system $\mathcal{L}^{\mathrm{LTS}}_{\mathsf{Kh}}$ (Table 1) is sound and strongly complete w.r.t. the class $\mathbf{M_{AC}}$.*

Proof. The arguments are exactly as in Theorem 5, using this time Proposition 13 and Proposition 14.

Summing up, while in Subsection 7.1 we asked for the agent to have all plans available and to be able to distinguish each one from each other, here we are a bit more general and ask for the agent to have sets of plans that can mimic the behaviour of those plans that a actually 'do something'.

8 Finite model property and complexity

This section is devoted to the study of the computational complexity of the logic $\mathsf{L}_{\mathsf{Kh}_i}$ over LTS^Us. To do so, we will use two standard tools from modal logic: filtration and selection (see, e.g., [10] for details). First, we define a notion of filtration that, given an arbitrary model and a formula, allows us to obtain a finite model that satisfies the formula if and only if the original model satisfies it. This proves that the satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i}$ is decidable. Then, we define a (more specialized) selection function which, from a canonical model, enables us to extract a polynomial-size model. Thus, we show that the satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i}$ is NP-complete (given that we provide a model checking algorithm running in P).

8.1 Finite model property via filtrations

We start by introducing two relations that will be crucial to define a proper notion of filtration, given a set of formulas Σ and a model $\mathcal{M}$.

Definition 8.1 (Σ-equivalence) Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathrm{S}_i\}_{i\in\mathsf{Agt}}, \mathrm{V}, \mathsf{Act}\rangle$ be an LTS^U and let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i}$-formulas closed under subformulas. Define the relations $\leftrightsquigarrow_\Sigma \subseteq \mathrm{W} \times \mathrm{W}$ and $\leftrightarrows_\Sigma \subseteq \mathrm{S}_{\mathsf{Agt}} \times \mathrm{S}_{\mathsf{Agt}}$ (with $\mathrm{S}_{\mathsf{Agt}} := \bigcup_{i\in\mathsf{Agt}} \mathrm{S}_i$) as:

$$w \leftrightsquigarrow_\Sigma v \quad \mathit{iff}_{def} \quad \text{for all } \psi \in \Sigma,\ \mathcal{M}, w \models \psi \text{ iff } \mathcal{M}, v \models \psi,$$

$$\pi \leftrightarrows_\Sigma \pi' \quad \mathit{iff}_{def} \quad \text{for all } i \in \mathsf{Agt} \text{ and } \mathsf{Kh}_i(\psi,\varphi) \in \Sigma,\ \pi \text{ is a witness for } \mathsf{Kh}_i(\psi,\varphi) \text{ in } \mathcal{M} \text{ iff } \pi' \text{ is a witness for } \mathsf{Kh}_i(\psi,\varphi) \text{ in } \mathcal{M}.$$

Notice that $\leftrightsquigarrow_\Sigma$ (a generalization of $\leftrightsquigarrow$ in Definition 5.3 to a given set of formulas) and $\leftrightarrows_\Sigma$ are equivalence relations over W and $\mathrm{S}_{\mathsf{Agt}}$, respectively. For $w \in \mathrm{W}$ (resp., $\pi \in 2^{(\mathsf{Act}^*)}$), we use $[w]_\Sigma$ (resp., $[\pi]_\Sigma$) to denote w's (resp., π's) Σ-equivalence class; i.e.,

$$[w]_\Sigma := \{v \in \mathrm{W} \mid w \leftrightsquigarrow_\Sigma v\}; \qquad [\pi]_\Sigma := \{\pi' \in 2^{(\mathsf{Act}^*)} \mid \pi \leftrightarrows_\Sigma \pi'\}.$$

Although the notation $[\_]_\Sigma$ is overloaded, its argument will always disambiguate its use.

Definition 8.2 Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathrm{S}_i\}_{i\in\mathsf{Agt}}, \mathrm{V}, \mathsf{Act}\rangle$ be an LTS^U and let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i}$-formulas that is closed under subformulas. For $i \in \mathsf{Agt}$ define $\mathsf{Act}_i^\Sigma := \{a_{[\pi]_\Sigma} \mid \pi \in \mathrm{S}_i \text{ is a witness of some } \mathsf{Kh}_i(\psi,\varphi) \in \Sigma \text{ in } \mathcal{M}\}$; and $\mathsf{Act}^\Sigma := \bigcup_{i\in\mathsf{Agt}} \mathsf{Act}_i^\Sigma$.

The idea behind the definition of Act^Σ is that, for each $\mathsf{Kh}_i(\psi,\varphi) \in \Sigma$ that is true at $\mathcal{M}$, we consider an action mimicking the behaviour of those sets of plans π that witness the satisfiability of $\mathsf{Kh}_i(\psi,\varphi)$ in $\mathcal{M}$.

Now, we are in position of defining the notion of filtration.

Definition 8.3 (Filtration of $\mathcal{M}$ through Σ) Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathrm{S}_i\}_{i\in\mathsf{Agt}}, \mathrm{V}, \mathsf{Act}\rangle$ be an LTS^U; let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i}$-formulas that is closed under subformulas. An LTS^U $\mathcal{M}^f = \langle \mathrm{W}^f, \mathrm{R}^f, \{\mathrm{S}_i^f\}_{i\in\mathsf{Agt}}, \mathrm{V}^f, \mathsf{Act}^\Sigma\rangle$ is *a filtration of $\mathcal{M}$ through Σ* if and only if it satisfies the following conditions:

(1) $\mathrm{W}^f := \{[w]_\Sigma \mid w \in \mathrm{W}\}$;

(2) $V^f([w]_\Sigma) := \{p \in \Sigma \mid \mathcal{M}, w \models p\}$;

(3) for all $i \in \mathsf{Agt}$, $a_{[\pi]_\Sigma} \in \mathsf{Act}_i^\Sigma$ implies $\{a_{[\pi]_\Sigma}\} \in S_i^f$;

(4) S_i^f is finite and well-defined (as per Remark 1), and each $\pi \in S_i^f$ is finite;

(5) for all $\pi \in S_i^f$, if π is a witness of some $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma$ in $\mathcal{M}^f$, then there is $\pi' \in S_i$ such that π' is a witness of $\mathsf{Kh}_i(\psi, \varphi)$ in $\mathcal{M}$;

(6) if $(w, v) \in R_\pi$ and $a_{[\pi]_\Sigma} \in \mathsf{Act}_i^\Sigma$, then $([w]_\Sigma, [v]_\Sigma) \in R^f_{a_{[\pi]_\Sigma}}$;

(7) if $([w]_\Sigma, [v]_\Sigma) \in R^f_{a_{[\pi]_\Sigma}}$, and π is a witness of some $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma$ in $\mathcal{M}$, then $w \in [\![\psi]\!]^{\mathcal{M}}$ implies $v \in [\![\varphi]\!]^{\mathcal{M}}$. ⊣

Note that V^f is well-defined: given $p \in \Sigma$, if $[w]_\Sigma = [v]_\Sigma$ and $\mathcal{M}, w \models p$, then $\mathcal{M}, v \models p$. Also, as S_i^f is well-defined (by definition), we have that $\mathcal{M}^f$ is an LTS^U over Prop and Agt. Also, note that if ***(5)***, ***(6)*** and ***(7)*** above are turned into if and only if conditions, they always define an LTS^U which is a filtration.

Definition 8.3 deserves further comments. Notice that, for the LTS part, the filtration is defined similarly as for the basic modal logic (see, e.g., [10]). The most significant difference is the change in the labelling of the relations, since we now use Act^Σ as the set of action names, instead of Act. But this has a consequence on the definition of S_i^f. The relation $\leftrightarrows_\Sigma$ enables us, from a finite set Σ, to obtain a finite set of witnesses for the formulas $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma$, from which we also get that Act^Σ and W^f are finite (for the latter we also use the definition of $\leftrightsquigarrow_\Sigma$). However, the new set S_i^f is defined in terms of a new set of action names, so there are potentially infinite new available plans to consider. Thus, we need to state that S_i^f is any finite set, satisfying the minimum and maximum conditions, whose members are also finite, and that is well-defined.

Theorem 7 *Let $\mathcal{M} = \langle W, R, \{S_i\}_{i \in \mathsf{Agt}}, V, \mathsf{Act}\rangle$ be an LTS^U and let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i}$-formulas that is closed under subformulas. Then, for all $\psi \in \Sigma$ and $w \in W$, $\mathcal{M}, w \models \psi$ iff $\mathcal{M}^f, [w]_\Sigma \models \psi$. Moreover, if Σ is finite then $\mathcal{M}^f$ is a finite model.*

Proof. Boolean cases work as expected. So, we will only show that $\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi)$ iff $\mathcal{M}^f, [w]_\Sigma \models \mathsf{Kh}_i(\psi, \varphi)$.

(⇒) Suppose that $\mathcal{M} \models \mathsf{Kh}_i(\psi, \varphi)$: let $\pi \in S_i$ be such that $[\![\psi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and $R_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$. By definition, $a_{[\pi]_\Sigma} \in \mathsf{Act}_i^\Sigma$ and therefore, $\{a_{[\pi]_\Sigma}\} \in S_i^f$. If $[\![\psi]\!]^{\mathcal{M}^f} = \varnothing$, the result trivially follows. Otherwise, let $[w]_\Sigma \in [\![\psi]\!]^{\mathcal{M}^f}$. By IH, $w \in [\![\psi]\!]^{\mathcal{M}}$, and since π is SE at w, $R_\pi(w) \neq \varnothing$. Since $\mathcal{M}^f$ is a filtration, we have that $R_{a_{[\pi]_\Sigma}}(w) \neq \varnothing$ and $\{a_{[\pi]_\Sigma}\}$ is SE at $[w]_\Sigma$. Thus, $[\![\psi]\!]^{\mathcal{M}^f} \subseteq \mathrm{SE}(\{a_{[\pi]_\Sigma}\})$.

Let $([w]_\Sigma, [v]_\Sigma) \in R^f_{a_{[\pi]_\Sigma}}$ be such that $[w]_\Sigma \in [\![\psi]\!]^{\mathcal{M}^f}$. By IH, $w \in [\![\psi]\!]^{\mathcal{M}}$. Since π is a witness of $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma$ in $\mathcal{M}$ (by assumption), by the definition of $\mathcal{M}^f$ we get $v \in [\![\varphi]\!]^{\mathcal{M}}$. Again, by IH, $[v]_\Sigma \in [\![\varphi]\!]^{\mathcal{M}^f}$. Thus, $R^f_{\{a_{[\pi]_\Sigma}\}}([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$. Therefore, $\mathcal{M}^f \models \mathsf{Kh}_i(\psi, \varphi)$.

(⇐) Suppose that $\mathcal{M}^f \models \mathsf{Kh}_i(\psi, \varphi)$: let $\pi \in S_i^f$ be such that $[\![\psi]\!]^{\mathcal{M}^f} \subseteq \mathrm{SE}(\pi)$ and $R_\pi([\![\psi]\!]^{\mathcal{M}^f}) \subseteq [\![\varphi]\!]^{\mathcal{M}^f}$. By definition of $\mathcal{M}^f$, since π is a witness of $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma$

in $\mathcal{M}^f$, we have that there is $\pi' \in \mathsf{S}_i$ such that π' is a witness of $\mathsf{Kh}_i(\psi, \varphi)$ in $\mathcal{M}$. Thus, $\mathcal{M} \models \mathsf{Kh}_i(\psi, \varphi)$.

It remains to show that $\mathcal{M}^f$ is finite. First, note that the number of elements in W^f is 2^m, with m being the number of formulas in Σ. By definition, for all $i \in \mathsf{Agt}$, Act_i^Σ is at most the number of $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma$, since if there are two groups of witnesses $[\pi]_\Sigma$ and $[\pi']_\Sigma$ for some $\mathsf{Kh}_i(\psi, \varphi)$, $[\pi]_\Sigma = [\pi']_\Sigma$. Hence, Act^Σ is polynomial in the number of $\mathsf{Kh}_i(\psi, \varphi)$ in Σ. Finally, by definition, S_i^f is finite. Thus, $\mathcal{M}^f$ is finite. ■

The last theorem states that every satisfiable formula of $\mathsf{L}_{\mathsf{Kh}_i}$, is satisfiable in a finite model. As a consequence, the satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i}$ is decidable. In the next section we will refine this result and provide exact complexity bounds.

8.2 Complexity via selection

Here we investigate the computational complexity of the satisfiability problem of $\mathsf{L}_{\mathsf{Kh}_i}$ under the LTS^U-based semantics. We will establish membership in NP by showing a polynomial-size model property.

Given a formula, we will show that it is possible to select just a piece of the canonical model which is relevant for its evaluation. The selected model will preserve satisfiability, and moreover, its size will be polynomial w.r.t. the size of the input formula.

Definition 8.4 (Selection function) Let $\mathcal{M}^\Gamma = \langle \mathrm{W}^\Gamma, \mathrm{R}^\Gamma, \{\mathrm{S}_i^\Gamma\}_{i \in \mathsf{Agt}}, \mathrm{V}^\Gamma, \mathsf{Act}^\Gamma \rangle$ be a canonical model for an MCS Γ (see Definition 6.1); take $w \in \mathrm{W}^\Gamma$ and a formula $\varphi \in \mathsf{L}_{\mathsf{Kh}_i}$. Define $\mathsf{Act}_\varphi := \{\langle \theta_1, \theta_2 \rangle \in \mathsf{Act}^\Gamma \mid \mathsf{Kh}_i(\theta_1, \theta_2)$ is a subformula of $\varphi\}$. A *canonical selection function* sel_w^φ is a function that takes $\mathcal{M}^\Gamma$, w and φ as input, returns a set $\mathrm{W}' \subseteq \mathrm{W}^\Gamma$, and is such that:

(1) $\mathsf{sel}_w^\varphi(p) = \{w\}$;

(2) $\mathsf{sel}_w^\varphi(\neg\varphi_1) = \mathsf{sel}_w^\varphi(\varphi_1)$

(3) $\mathsf{sel}_w^\varphi(\varphi_1 \vee \varphi_2) = \mathsf{sel}_w^\varphi(\varphi_1) \cup \mathsf{sel}_w^\varphi(\varphi_2)$;

(4) If $[\![\mathsf{Kh}_i(\varphi_1, \varphi_2)]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$ and $[\![\varphi_1]\!]^{\mathcal{M}^\Gamma} = \emptyset$: $\mathsf{sel}_w^\varphi(\mathsf{Kh}_i(\varphi_1, \varphi_2)) = \{w\}$;

(5) If $[\![\mathsf{Kh}_i(\varphi_1, \varphi_2)]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$ and $[\![\varphi_1]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$:
$\mathsf{sel}_w^\varphi(\mathsf{Kh}_i(\varphi_1, \varphi_2)) = \{w_1, w_2\} \cup \mathsf{sel}_{w_1}^\varphi(\varphi_1) \cup \mathsf{sel}_{w_2}^\varphi(\varphi_2)$, where w_1, w_2 are s.t. $(w_1, w_2) \in \mathrm{R}^\Gamma_{\langle \varphi_1, \varphi_2 \rangle}$;

(6) If $[\![\mathsf{Kh}_i(\varphi_1, \varphi_2)]\!]^{\mathcal{M}^\Gamma} = \emptyset$ (note that $[\![\varphi_1]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$):
For all set of plans π, either $[\![\varphi_1]\!]^{\mathcal{M}^\Gamma} \not\subseteq \mathrm{SE}(\pi)$ or $\mathrm{R}^\Gamma_\pi([\![\varphi_1]\!]^{\mathcal{M}^\Gamma}) \not\subseteq [\![\varphi_2]\!]^{\mathcal{M}^\Gamma}$. For each $a \in \mathsf{Act}_\varphi$:

  (a) if $[\![\varphi_1]\!]^{\mathcal{M}^\Gamma} \not\subseteq \mathrm{SE}(\{a\})$: we add $\{w_1\} \cup \mathsf{sel}_{w_1}^\varphi(\varphi_1)$ to $\mathsf{sel}_w^\varphi(\mathsf{Kh}_i(\varphi_1, \varphi_2))$, where $w_1 \in [\![\varphi_1]\!]^{\mathcal{M}^\Gamma}$ and $w_1 \notin \mathrm{SE}(\{a\})$;

(b) if $R^\Gamma_\pi([\![\varphi_1]\!]^{\mathcal{M}^\Gamma}) \not\subseteq [\![\varphi_2]\!]^{\mathcal{M}^\Gamma}$ we add $\{w_1, w_2\} \cup \mathsf{sel}^\varphi_{w_1}(\varphi_1) \cup \mathsf{sel}^\varphi_{w_2}(\varphi_2)$ to $\mathsf{sel}^\varphi_w(\mathsf{Kh}_i(\varphi_1, \varphi_2))$, where $w_1 \in [\![\varphi_1]\!]^{\mathcal{M}^\Gamma}$, $w_2 \in R^\Gamma_a(w_1)$ and $w_2 \notin [\![\varphi_2]\!]^{\mathcal{M}^\Gamma}$. ⊣

It is worth noticing that the case $[\![\mathsf{Kh}_i(\varphi_1, \varphi_2)]\!]^{\mathcal{M}^\Gamma} = \varnothing$ and $[\![\varphi_1]\!]^{\mathcal{M}^\Gamma} = \varnothing$ is not treated, as this is an impossible situation: φ_1 unsatisfiable makes $\mathsf{Kh}_i(\varphi_1, \varphi_2)$ trivially true, since the set S_i is not empty.

We can now use the function just defined, to select a small model which preserves the satisfiability of a given formula.

Definition 8.5 (Selected model) Let $\mathcal{M}^\Gamma$ be the canonical model for an MCS Γ, w a state in $\mathcal{M}^\Gamma$, and φ an $\mathsf{L}_{\mathsf{Kh}_i}$-formula. Let sel^φ_w be a selection function, we define the *model selected by* sel^φ_w as $\mathcal{M}^\varphi_w = \langle W^\varphi_w, R^\varphi_w, \{(S^\varphi_w)_i\}_{i \in \mathsf{Agt}}, V^\varphi_w, \mathsf{Act}^\varphi_w \rangle$, where

- $W^\varphi_w := \mathsf{sel}^\varphi_w(\varphi)$;
- $\mathsf{Act}^\varphi_w := \mathsf{Act}_\varphi \cup \{\langle \bot, \top \rangle\}$;
- $(R^\varphi_w)_{\langle \theta_1, \theta_2 \rangle} := R^\Gamma_{\langle \theta_1, \theta_2 \rangle} \cap (W^\varphi_w)^2$ for each $\langle \theta_1, \theta_2 \rangle \in \mathsf{Act}_\varphi$ and $(R^\varphi_w)_{\langle \bot, \top \rangle} := \varnothing$;
- $(S^\varphi_w)_i := \{\{a\} \mid a \in \mathsf{Act}_\varphi\}$, for $i \in \mathsf{Agt}$;
- V^φ_w is the restriction of V^Γ to W^φ_w. ⊣

Note that, although Act_φ can be an empty set, Act^φ_w and each collection of sets of plans $(S^\varphi_w)_i$ are not. Therefore, $\mathcal{M}^\varphi_w$ is an LTS^U.

Proposition 15 *Let $\mathcal{M}^\Gamma$ be a canonical model, w a state in $\mathcal{M}^\Gamma$ and φ an $\mathsf{L}_{\mathsf{Kh}_i}$-formula. Let $\mathcal{M}^\varphi_w$ be the selected model by a selection function sel^φ_w. Then, $\mathcal{M}^\Gamma, w \models \varphi$ implies that for all ψ subformula of φ, and for all $v \in W^\varphi_w$ we have that $\mathcal{M}^\Gamma, v \models \psi$ iff $\mathcal{M}^\varphi_w, v \models \psi$. Moreover, $\mathcal{M}^\varphi_w$ is polynomial on the size of φ.*

Proof. The proof proceeds by induction in the size of the formula:

Case $\psi = p$: if $\mathcal{M}^\Gamma, v \models p$, then $p \in V^\Gamma(v)$. Given that $v \in W^\varphi_w$, we have $p \in V^\varphi_w(v)$ and therefore $\mathcal{M}^\varphi_w, v \models p$. The other direction is similar.

Case $\psi = \neg\psi_1$: if $\mathcal{M}^\Gamma, w \models \neg\psi_1$, then $\mathcal{M}^\Gamma, w \not\models \psi_1$. By IH, $\mathcal{M}^\varphi_w, w \not\models \psi_1$ and therefore $\mathcal{M}^\varphi_w, w \models \neg\psi_1$. The other direction is similar.

Case $\psi = \psi_1 \vee \psi_2$: if $\mathcal{M}^\Gamma, v \models \psi_1 \vee \psi_2$, then $\mathcal{M}^\Gamma, v \models \psi_1$ or $\mathcal{M}^\Gamma, v \models \psi_2$. By IH, $\mathcal{M}^\varphi_w, v \models \psi_1$ or $\mathcal{M}^\varphi_w, v \models \psi_2$ and therefore $\mathcal{M}^\varphi_w, v \models \psi_1 \vee \psi_2$. The other direction is similar.

Case $\psi = \mathsf{Kh}_i(\psi_1, \psi_2)$: Suppose that $\mathcal{M}^\Gamma, v \models \mathsf{Kh}_i(\psi_1, \psi_2)$. We consider two possibilities:

- $\boldsymbol{[\![\psi_1]\!]^{\mathcal{M}^\Gamma} = \varnothing}$. Since $\mathcal{M}^\Gamma, v \models \mathsf{Kh}_i(\psi_1, \psi_2)$ there is a $\pi \in S^\Gamma_i$ s.t. $\varnothing = [\![\psi_1]\!]^{\mathcal{M}^\Gamma} \subseteq SE^{\mathcal{M}^\Gamma}(\pi)$ and $\varnothing = R^\Gamma_\pi([\![\psi_1]\!]^{\mathcal{M}^\Gamma}) \subseteq [\![\psi_2]\!]^{\mathcal{M}^\Gamma}$. By IH $[\![\psi_1]\!]^{\mathcal{M}^\varphi_w} \subseteq [\![\psi_1]\!]^{\mathcal{M}^\Gamma}$. Notice that, since $[\![\psi_1]\!]^{\mathcal{M}^\Gamma} = \varnothing$, we also have $[\![\psi_1]\!]^{\mathcal{M}^\varphi_w} = \varnothing$. Let $\pi' = \{\langle \bot, \top \rangle\}$, we know that $\pi' \in (S^\varphi_w)_i$, and $(R^\varphi_w)_{\pi'}([\![\psi_1]\!]^{\mathcal{M}^\varphi_w}) = \varnothing$. So, there is a $\pi' \in (S^\varphi_w)_i$ s.t. $[\![\psi_1]\!]^{\mathcal{M}^\varphi_w} \subseteq SE^{\mathcal{M}^\varphi_w}(\pi')$ and $(R^\varphi_w)_{\pi'}([\![\psi_1]\!]^{\mathcal{M}^\varphi_w}) \subseteq [\![\psi_2]\!]^{\mathcal{M}}$. Therefore, $\mathcal{M}^\varphi_w, v \models \mathsf{Kh}_i(\psi_1, \psi_2)$.

- $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \neq \varnothing$: since $\mathcal{M}^\Gamma, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$ there exists a $\pi \in \mathrm{S}^\Gamma_i$ s.t. $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \subseteq \mathrm{SE}^{\mathcal{M}^\Gamma}(\pi)$ and $\mathrm{R}^\Gamma_\pi(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. By Truth Lemma, $\mathsf{Kh}_i(\psi_1,\psi_2) \in v$, then $\mathsf{Kh}_i(\psi_1,\psi_2) \in \Gamma$ and $\langle\psi_1,\psi_2\rangle \in \mathsf{Act}^\Gamma$. By the definition of $\mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}$, we have that for all $w \in \llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}$, it holds that $\mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(w) \neq \varnothing$ and $\mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(w) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Thus, $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \subseteq \mathrm{SE}^{\mathcal{M}^\Gamma}(\{\langle\psi_1,\psi_2\rangle\})$ and $\mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Since $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \neq \varnothing$, there exist $w_1, w_2 \in \mathrm{W}^\Gamma$ s.t. $(w_1,w_2) \in \mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}$.

  Notice that by definition of $\mathcal{M}^\varphi_w$, we have that $\{\langle\psi_1,\psi_2\rangle\} \in (\mathrm{S}^\varphi_w)_i$ and that $(\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}$ is defined. Also, by the definition of sel^φ_w, Item *(5)*, there exist $w'_1, w'_2 \in \mathrm{W}^\varphi_w$ s.t. $(w'_1,w'_2) \in (\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}$. Let $v_1 \in \llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \subseteq \llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}$ (the inclusion holds by IH). Then, we have $v_1 \in \mathrm{SE}^{\mathcal{M}^\Gamma}(\{\langle\psi_1,\psi_2\rangle\})$ and $\mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(v_1) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Since for all $v_2 \in \mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(v_1)$, we have $v_2 \in \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$, (in particular $v_2 = w'_2$), then $w'_2 \in (\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}(v_1)$. Thus, $v_1 \in \mathrm{SE}^{\mathcal{M}^\varphi_w}(\{\langle\psi_1,\psi_2\rangle\})$.

  Aiming for a contradiction, suppose now that $(\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}(v_1) = \mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(v_1) \cap \mathrm{W}^\varphi_w \not\subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\varphi_w}$; and let $v_2 \in (\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}(v_1)$ s.t. $v_2 \notin \llbracket\psi_2\rrbracket^{\mathcal{M}^\varphi_w}$. Then we have that $(\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}(v_1) \subseteq \mathrm{R}^\Gamma_{\langle\psi_1,\psi_2\rangle}(v_1)$, but also by IH $v_2 \notin \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Thus, $\{\langle\psi_1,\psi_2\rangle\}$ is not a witness for $\mathsf{Kh}_i(\psi_1,\psi_2)$ in $\mathcal{M}^\Gamma$, which is a contradiction. Then, it must be the case that $(\mathrm{R}^\varphi_w)_{\{\langle\psi_1,\psi_2\rangle\}}(v_1) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\varphi_w}$. Since we showed that $\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \subseteq \mathrm{SE}^{\mathcal{M}^\varphi_w}(\{\langle\psi_1,\psi_2\rangle\})$ and $(\mathrm{R}^\varphi_w)_{\{\langle\psi_1,\psi_2\rangle\}}(\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w}) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\varphi_w}$, we conclude $\mathcal{M}^\varphi_w, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$.

Assume now $\mathcal{M}^\varphi_w, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$. Again, we consider two possibilities:

- $\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} = \varnothing$: since $\mathcal{M}^\varphi_w, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$, then $\varnothing = \llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \subseteq \mathrm{SE}^{\mathcal{M}^\varphi_w}(\pi')$ and $\varnothing = (\mathrm{R}^\varphi_w)_{\pi'}(\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w}) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\varphi_w}$ for some $\pi' \in (\mathrm{S}^\varphi_w)_i$. We claim that $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} = \varnothing$. Because otherwise if $\mathcal{M}^\Gamma, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$, by sel^φ_w, Item *(5)*, $\varnothing \neq (\mathrm{R}^\varphi_w)_{\langle\psi_1,\psi_2\rangle}$ is defined and $\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \neq \varnothing$, contradicting hypothesis. And if $\mathcal{M}^\Gamma, v \not\models \mathsf{Kh}_i(\psi_1,\psi_2)$, by sel^φ_w, item Item *(6)*, and IH, $\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \neq \varnothing$, again a contradiction.

  Let π be any set of plans in S^Γ_i; since $\mathrm{R}^\Gamma_\pi(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) = \varnothing$, $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}^\Gamma_\pi(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Then, $\mathcal{M}^\Gamma, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$.

- $\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \neq \varnothing$: first, notice that by IH, $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \neq \varnothing$. Also, by $\mathcal{M}^\varphi_w, v \models \mathsf{Kh}_i(\psi_1,\psi_2)$, we get $\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \subseteq \mathrm{SE}^{\mathcal{M}^\varphi_w}(\pi')$ and $(\mathrm{R}^\varphi_w)_{\pi'}(\llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w}) \subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\varphi_w}$, for some $\pi' \in (\mathrm{S}^\varphi_w)_i$. Aiming for a contradiction, suppose $\mathcal{M}^\Gamma, v \not\models \mathsf{Kh}_i(\psi_1,\psi_2)$. This implies that for all $\pi \in \mathrm{S}^\Gamma_i$, $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \not\subseteq \mathrm{SE}^{\mathcal{M}^\Gamma}(\pi)$ or $\mathrm{R}^\Gamma_\pi(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) \not\subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Also, by definition of Act_φ we have that for all $\pi = \{a\} \in (\mathrm{S}^\varphi_w)_i$, with $a \in \mathsf{Act}_\varphi$, $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \not\subseteq \mathrm{SE}^{\mathcal{M}^\Gamma}(\pi)$ or $\mathrm{R}^\Gamma_\pi(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) \not\subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$; i.e., for all $a \in \mathsf{Act}_\varphi$ $\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma} \not\subseteq \mathrm{SE}^{\mathcal{M}^\Gamma}(\{a\})$ or $\mathrm{R}^\Gamma_{\{a\}}(\llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}) \not\subseteq \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. Thus, there exists $w_1 \in \llbracket\psi_1\rrbracket^{\mathcal{M}^\Gamma}$ s.t. $w_1 \notin \mathrm{SE}^{\mathcal{M}^\Gamma}(a)$ or there exists $w_2 \in \mathrm{R}^\Gamma_a(w_1)$ s.t. $w_2 \notin \llbracket\psi_2\rrbracket^{\mathcal{M}^\Gamma}$. By definition of sel^φ_w, Item *(6)*, we add witnesses for each $a \in \mathsf{Act}_\varphi$. So, let $\pi' \in (\mathrm{S}^\varphi_w)_i$. If $\pi' = \{\langle\bot,\top\rangle\}$, trivially we obtain $\varnothing \neq \llbracket\psi_1\rrbracket^{\mathcal{M}^\varphi_w} \not\subseteq \mathrm{SE}^{\mathcal{M}^\varphi_w}(\pi') = \varnothing$. Then, take another $\pi' = \{a\}$ s.t. $a \in \mathsf{Act}_\varphi$, and $w_1' \in [\![\psi_1]\!]^{\mathcal{M}_w^\varphi} \subseteq [\![\psi_1]\!]^{\mathcal{M}^\Gamma}$. If $w_1' \notin \mathrm{SE}^{\mathcal{M}^\Gamma}(\{a\})$, $\mathrm{R}_a^\Gamma(w_1') = \varnothing$ and thus $(\mathrm{R}_w^\varphi)_a(w_1') = \varnothing$ and therefore $w_1' \notin \mathrm{SE}^{\mathcal{M}_w^\varphi}(\{a\})$. On the other hand, if there exists $w_2 \in \mathrm{R}_a^\Gamma(w_1')$ s.t. $w_2 \notin [\![\psi_2]\!]^{\mathcal{M}^\Gamma}$, then by sel_w^φ and IH, there exists $w_2' \in \mathrm{W}_w^\varphi$ s.t. $w_2' \in \mathrm{R}_a^\Gamma(w_1')$ and $w_2' \notin [\![\psi_2]\!]^{\mathcal{M}_w^\varphi}$, and consequently, there exists $w_2' \in (\mathrm{R}_w^\varphi)_a(w_1')$ s.t. $w_2' \notin [\![\psi_2]\!]^{\mathcal{M}_w^\varphi}$. In any case, it leads to $\mathcal{M}_w^\varphi, v \not\models \mathsf{Kh}_i(\psi_1, \psi_2)$, a contradiction. Therefore, $\mathcal{M}^\Gamma, v \models \mathsf{Kh}_i(\psi_1, \psi_2)$.

Notice now that the selection function adds states from $\mathcal{M}^\Gamma$, only for each Kh_i-formula that appears as a subformula of φ; and the number of states added at each time is polynomial in $|\varphi|$. Hence, the size of W_w^φ is polynomial. Since $(\mathrm{S}_w^\varphi)_i$ is also polynomial, the size of $\mathcal{M}_w^\varphi$ is polynomial in $|\varphi|$. ■

In order to prove that the satisfiability problem of $\mathsf{L}_{\mathsf{Kh}_i}$ is in NP, it remains to show that the model checking problem is in P.

Proposition 16 *The model checking problem for* $\mathsf{L}_{\mathsf{Kh}_i}$ *is in* P.

Proof. Given a pointed LTS^U $\mathcal{M}, w$ and a formula φ, we define a bottom-up labeling algorithm running in polynomial time which checks whether $\mathcal{M}, w \models \varphi$. We follow the same ideas as for the basic modal logic K (see e.g., [11]). Below we introduce the case for formulas of the shape $\mathsf{Kh}_i(\psi, \varphi)$, over an LTS^U $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathrm{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V}, \mathsf{Act} \rangle$:

```
Procedure ModelChecking((M, w), Kh_i(ψ, φ))
lab(Kh_i(ψ, φ)) ← ∅;
for all π ∈ S_i do
    kh ← True;
    for all σ ∈ π do
        for all v ∈ lab(ψ) do
            kh ← (kh & v ∈ SE(σ) & R_σ(v) ⊆ lab(φ));
        end for
    end for
    if kh then
        lab(Kh_i(ψ, φ)) ← W;
    end if
end for
```

As S_i and each $\pi \in \mathrm{S}_i$ are not empty, the first two **for** loops are necessarily executed. If $lab(\psi) = \varnothing$, then the formula $\mathsf{Kh}_i(\psi, \varphi)$ is trivially true. Otherwise, *kh* will remain true only if the appropriate conditions for the satisfiability of $\mathsf{Kh}_i(\psi, \varphi))$ hold. If no π succeeds, then the initialization of $lab(\mathsf{Kh}_i(\psi, \varphi))$ as $\varnothing$ will not be overwritten, as it should be. Both $v \in \mathrm{SE}(\sigma)$ and R_σ can be verified in polynomial time. Hence, the model checking problem is in P. ■

The intended result for satisfiability now follows.

Theorem 8 *The satisfiability problem for* $\mathsf{L}_{\mathsf{Kh}_i}$ *over* LTS^U*s is* NP*-complete.*

Proof. Hardness follows from NP-completeness of propositional logic (a fragment of $\mathsf{L}_{\mathsf{Kh}_i}$). By Proposition 15, each satisfiable formula φ has a model of polynomial-size on φ. Thus, we can guess a polynomial model $\mathcal{M}, w$, and verify $\mathcal{M}, w \models \varphi$ (which can be done in polyonomial time, due to Proposition 16). Thus, the satisfiability problem is in the class NP. ■

9 Final remarks

In this article, we introduce a new semantics for the *knowing how* modality from [58, 59, 60] (for multiple agents), defined in terms of *uncertainty-based labeled transition systems (*LTS^U*)*. The novelty in our proposal is that LTS^U*s* are equipped with an indistinguishability relation among plans. In this way, the epistemic notion of uncertainty of an agent –which in turn defines her epistemic state– is reintroduced, bringing the notion of *knowing how* closer to the notion of *knowing that* from classical epistemic logics. We believe that the semantics based on LTS^U can represent properly the situation of a shared, objective description of the affordances of a given situation, together with the different, subjective and personal abilities of a group of agents. This seems difficult to achieve using a semantics based on LTSs alone.

We show that the logic of [58, 59, 60] can be obtained by imposing particular conditions over LTS^U; thus, the new semantics is more general. In particular, it provides counter-examples to $\mathcal{EMP}$ and $\mathcal{COMP}$, which directly link the knowing how modality Kh to properties of the universal modality. Indeed, consider $\mathcal{EMP}$: even though $\mathsf{A}(\psi \rightarrow \varphi)$ objectively holds in the underlying LTS of an LTS^U, it could be argued that an agent might not be aware of actions or plans to turn those facts into knowledge, resulting in $\mathsf{Kh}(\psi, \varphi)$ failing in the model.

To characterize validities in this language over LTS^Us, we introduce a sound and strongly complete axiom system.

We also define a suitable notion of bisimulation over LTS^U*s*, following ideas introduced in [18, 19]. We show that bisimilarity implies formula equivalence, and that finite models form a Hennessy-Milner class (i.e., that formula equivalence implies bisimilarity over finite models).

Finally, we prove that the satisfiability problem for our multi-agent knowing how logic over the LTS^U-based semantics is NP-complete. The proof relies on a selection argument on the canonical model, and on the fact that the model checking problem is polynomial. We also provide a filtration technique that, given an arbitrary model satisfying φ, returns a finite model that satisfies φ.

Future work. There are several interesting lines of research to explore in the future. First, our framework easily accommodates other notions of executability. For instance, one could require only some of the plans in a set π to be strongly executable, or weaken the condition of *strong* executability, and so on. We can also explore the effects of imposing different restrictions on the construction of the indistinguishability relation between plans. More precisely, we would like to look into specific definitions of the uncertainty relation, to study potential additional properties of the logic of such agents. A natural case is when the uncertainty over plans arises from a 'more basic' uncertainty over basic actions, but one can think of more 'complex' forms of uncertainty, including cases in which the resulting relation is not of equivalence. It would be interesting to investigate which logics we obtain in these cases, and their relations with the LTS semantics.

Second, to our knowledge, the exact complexity of the satisfiability problem for knowing how over LTSs is open. A recent work [3] establishes that this problem is decidable in Σ_2^P, the second level in the polynomial heriarchy, but it would be interesting to obtain tight bounds for it.

Third, the LTS^U semantics, in the multi-agent setting, leads to natural definitions of concepts of *collective* knowing how, in the spirit of [14]. For instance, one can easily define a notion of *general knowing how* as $\mathsf{EKh}_G(\psi, \varphi) := \bigwedge_{i \in G} \mathsf{Kh}_i(\psi, \varphi)$, whose reading is *"everyone in the group G knows how to achieve φ given ψ"*; and *"somebody in the group G knows how to achieve φ given ψ"*, as $\mathsf{SKh}_G(\psi, \varphi) := \bigvee_{i \in G} \mathsf{Kh}_i(\psi, \varphi)$ (see, e.g., [1] for a similar approach in standard epistemic logic). Other, more complex notions such as *distributed* and *common knowing how*, deserve further exploration.

Finally, dynamic modalities capturing epistemic updates can be defined via operations that modify the indistinguishability relation among plans (as is done with other dynamic epistemic operators, see, e.g., [54]). This would allow to express different forms of communication, such as *public*, *private* and *semi-private* announcements concerning (sets of) plans. Some preliminary results have been presented in [5].

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