# Dynamic Epistemic Logics of Introspection

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# Motivation

- Epistemic Logics deal with the knowledge of agents.
- Not only about propositional facts (ontic knowledge), but with the knowledge about her own and/or other agents' knowledge (high-order knowledge).
- Two important notions in high-order knowledge: positive and negative introspection.
- Positive introspection: *"if the agent knows something, then she knows that she knows it"*.
- Negative introspection: *"if the agent does not know something, then she knows that she does not know it"*.

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  - More....

# Our work

- We work in a single-agent framework.
- Our approach is provide operations that after being executed, $+/$- introspection is achieved.
- Two kind of operations (both in a global sense):
  - General introspection.
  - Introspection with respect to a formula.
- Two possible approaches:
  - Pessimistic: the agent loses some knowledge.
  - Optimistic: the agent gains some knowledge.

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- Let $M = \langle W, R, V \rangle$ be a relational model, $w \in W$:

$$M, w \Vdash \langle + \rangle \varphi \quad \text{iff} \quad M^+, w \Vdash \varphi$$

with $M^+ = \langle W, R^+, V \rangle$, where $R^+$ is the transitive closure of $R$.

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**Proposition**

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- However, is not the expected behaviour:

**Fact**

$$\not\vdash \Box p \rightarrow [+] (\Box p \wedge \Box \Box p)$$

# Losing knowledge

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**Observation**

We are getting introspection at cost of losing knowledge!

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- We need to remove edges.

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$$(M, w) \Vdash \langle +'\chi\rangle\varphi \quad \text{iff} \quad (M_{+[\![\chi]\!]^M}, w) \Vdash \varphi$$

- But... this operation can take place in any situation! (not only when the agent knows $\chi$)

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*"The agent can perform a particular positive introspection step for $\chi$ after which she will know $\varphi$ iff she knows both $\chi$ and that, after the 'preconditionless' operation, $\varphi$ will be the case."*

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Replacing $\varphi$ by $\Box\chi$ we get:

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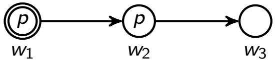
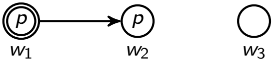



$M_{+[\![\chi]\!]^M}$ :



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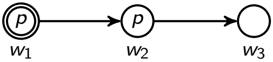


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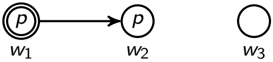


**Proposition**

If $\Vdash \chi \rightarrow [+'\chi]\chi$, then after the operation the agent will have positive introspection about $\chi$, i.e., $\Vdash \langle +\chi\rangle\Box\Box\chi \leftrightarrow \Box\chi$.

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$$\Vdash [-] (\neg \Box \varphi \to \Box \neg \Box \varphi).$$

Even more:

**Proposition**

If $\varphi$ is propositional, then $\Vdash \neg \Box \varphi \to [-] (\neg \Box \varphi \land \Box \neg \Box \varphi)$.

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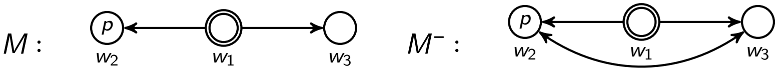

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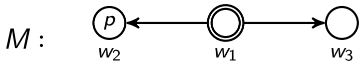


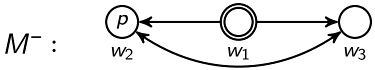


- Axiomatizable in PDL (with test and converse).

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$$R' := R \cup (\reflectbox{R} \circ (R \cup \reflectbox{R})^{*} \circ R \circ \mathsf{Id}_U^M),$$

where $\mathsf{Id}_U^M := \{(u, u) \mid u \in U\}$.

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- For the propositional case, the general operation already behaves as expected.
- However, for uniformity we also define the particular version.
- Let $M = \langle W, R, V \rangle$ and $U \subseteq W$. The $U$-connecting operation gives the model $M_{-U} = \langle W, R', V \rangle$ with

$$R' := R \cup (\reflectbox{R} \circ (R \cup \reflectbox{R})^* \circ R \circ \mathsf{Id}^M_U),$$

  where $\mathsf{Id}^M_U := \{(u, u) \mid u \in U\}$.
- We define a pre-conditionless operation:

$$(M, w) \Vdash \langle -'\chi \rangle \varphi \quad \text{iff} \quad (M_{-[\![\neg\chi]\!]^M}, w) \Vdash \varphi.$$

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- And the corresponding modality:

$$\langle -\chi \rangle \varphi := \neg \Box \chi \wedge \langle -'\chi \rangle \varphi.$$

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- We studied their properties, and obtained complete axiomatizations (via reductions axioms into PDL).
- For positive introspection: one operation adds edges, the other removes edges.
- For negative introspection both operations adds edges.

# Future Work

- Getting introspection in one level (more local): from $\Box p \wedge \neg \Box \Box p$ to $\Box p \wedge \Box \Box p \wedge \neg \Box \Box \Box p$
- We would like to explore this kind of operations in a multi-agent setting.
  - Public, private versions of these operations.
  - Reaching common knowledge.
- Connections with cognitive operations.