A Tableaux Calculus for Default Intuitionistic Logic

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$$\sout{\neg\neg\varphi \supset \varphi} \quad (\neg\neg \text{ elim.})$$
$$\sout{(\neg\varphi \supset \neg\psi) \supset (\psi \supset \varphi)} \quad \text{(contrapositive)}$$
$$\sout{((\varphi \supset \chi) \supset \varphi) \supset \varphi} \quad \text{(Peirce's Law)}$$

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Applications:
common sense reasoning – knowledge representation– software engineering– computer science – legal reasoning – planning

The Power of Intuitionistic + Default Reasoning I/II

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- Verdict of guilty $\boxed{g}$: prosecution presents evidence with the *"beyond reasonable doubt"* standard of proof.
- Verdict of not guilty $\boxed{\neg g}$: the defense manages to pinpoint *contradiction* in the evidences.
- This behaviour is intuitionistic: $\boxed{g \vee \neg g}$ is not plainly true.

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- The the PPI behaves non-monotonically.

Intuitionistic Propositional Logic in a nutshell

Syntax of Intuitionistic Propositional Logic (IPL):

$$\varphi ::= p_i \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \varphi \supset \varphi.$$

Models: a model is a tuple $\mathfrak{M} = \langle W, \preccurlyeq, V \rangle$, where

- W is a set of elements or worlds;
- $\preccurlyeq \subseteq W^2$ is *reflexive* and *transitive*; and
- $V : \mathrm{Prop} \to 2^W$ is s.t. for all $w \preccurlyeq w'$, if $w \in V(p)$, $w' \in V(p)$.

Semantics of IPL:

$$\begin{array}{lll} \mathfrak{M}, w \models \neg\varphi & \text{iff} & \text{for all } w \preccurlyeq w',\ \mathfrak{M}, w' \not\models \varphi \\ \mathfrak{M}, w \models \varphi \supset \psi & \text{iff} & \text{for all } w \preccurlyeq w',\ \text{if } \mathfrak{M}, w' \models \varphi \text{ then } \mathfrak{M}, w' \models \psi. \end{array}$$

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In our case, Φ, π, ρ, χ and φ, are formulas from intuitionistic logic.

Default Logical Consequence

Definition.

An extension E of a default theory $\langle \Phi, \Delta \rangle$, is a set $E = \text{Conseq}(\Phi \cup \{\chi \mid \pi \overset{\rho}{\Rightarrow} \chi \in \Delta'\})$, where $\Delta' \subseteq \Delta$.

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Definition (Default Consequence).

$\boxed{\langle \Phi, \Delta \rangle \approx \varphi}$, iff for all extension E, $\boxed{E \models \varphi}$.

Notice that $\models$ is the underlying consequence (in our case, IPL).

Tableaux Proof Calculus

We define a tableaux-based notion of default proof $\mathrel{|\!\sim}$, in correspondence with $\mathrel{|\!\approx}$.

The tableaux calculus is an extension of the calculus for IPL.

Tableaux formulas:

- $@_i^+ \varphi$ stands for "*φ holds at world i*";
- $@_i^- \varphi$ stands for "*φ does not hold at world i*";
- (i, j) stands for "*world j is accessible from world i*"

Tableaux calculus for IPL

The calculus decides logical consequence, i.e., let Φ a set of formulas, and φ a formula, it decides whether $\Phi \vDash \varphi$.

$$
\frac{@_i^+(\varphi \wedge \psi)}{\begin{array}{c} @_i^+\psi \\ @_i^+\varphi \end{array}}\ (\wedge^+)
\qquad
\frac{@_i^-(\varphi \wedge \psi)}{@_i^-\varphi \quad @_i^-\psi}\ (\wedge^-)
\qquad
\frac{@_i^+(\varphi \vee \psi)}{@_i^+\varphi \quad @_i^+\psi}\ (\vee^+)
\qquad
\frac{@_i^-(\varphi \vee \psi)}{\begin{array}{c} @_i^-\psi \\ @_i^-\varphi \end{array}}\ (\vee^-)
$$

$$
\frac{\begin{array}{c} @_i^+(\varphi \supset \psi) \\ (i,j) \end{array}}{@_j^-\varphi \quad @_j^+\psi}\ (\supset^+)
\qquad
\frac{@_i^-(\varphi \supset \psi)}{\begin{array}{c} (i,j) \\ @_j^+\varphi \\ @_j^-\psi \end{array}}\ (\supset^-)^\dagger
\qquad
\frac{\begin{array}{c} @_i^+\neg\varphi \\ (i,j) \end{array}}{@_j^-\varphi}\ (\neg^+)
\qquad
\frac{@_i^-\neg\varphi}{\begin{array}{c} (i,j) \\ @_j^+\varphi \end{array}}\ (\neg^-)^\dagger
$$

$$
\frac{\begin{array}{c} @_i^+ p \\ (i,j) \end{array}}{@_j^+ p}\ (\mathsf{her})^\ddagger
\qquad
\frac{}{(i,i)}\ (\mathsf{ref})^*
\qquad
\frac{\begin{array}{c} (i,j) \\ (j,k) \end{array}}{(i,k)}\ (\mathsf{trans})^\P
\qquad
\frac{}{@_0^+\varphi}\ (\mathsf{A}) \quad \text{for } \varphi \in \varPhi
$$

† for j new (i.e., not used before in the branch).
‡ for $j \neq i$ in the branch.
* for i in the branch.
¶ for i, j, k in the branch.

Tableaux calculus for $\mathcal{D}$IPL

The calculus decides default consequence, i.e., let $\langle \Phi, \Delta \rangle$ a default theory, and φ a formula, it decides whether $\langle \Phi, \Delta \rangle \approx \varphi$.

$$\frac{}{\begin{array}{ccccc} \delta_1 & & \delta_i & & \delta_n \\ @_0^+ \delta_1^{\mathrm{X}} & \ldots & @_0^+ \delta_i^{\mathrm{X}} & \ldots & @_0^+ \delta_n^{\mathrm{X}} \end{array}} \ (\mathsf{D})^{\dagger}$$

for $\{\, \delta_i \mid i \in [1, n] \,\} = \{\, \delta \in \Delta_\Theta \backslash \Delta_B \mid \delta$ is detached by $\Delta_B \,\}$
where Δ_B is the set of defaults in the branch.

Adequateness of the calculus

Theorem.
The calculus is sound, complete, and it terminates (by using loop-checks).

DefTab: a tableaux-based prover for $\mathcal{D}$IPL

- A prototype implementation in Haskell.
- Given $\langle \Phi, \Delta \rangle$ and φ as input, DefTab builds proof attempts of $\langle \Phi, \Delta \rangle \mathrel{|\!\!\sim} \varphi$ by searching for Kripke models for φ.
- Then it uses sentences from Φ and defaults from Δ.
- DefTab reports whether or not a default proof has been found.
- In the latter case, DefTab exhibits an extension of $\langle \Phi, \Delta \rangle$ from which φ does not follow.

Empirical evaluation

- We compare DefTab against intuitionistic provers:
  - **intuit:** SMT reasoner over MiniSAT;
  - **IntHistGC:** sequent based prover (with backtracking optimizations);
  - **fCube:** tableaux-based (specialized rules for nested implications).
  - These provers outperform DefTab (but comparable mostly in non-valid formulas).
  - Expected since DefTab does not implement optimizations yet.
- For the default part, we tested with non-trivial intuitionistic formulas, defaults do not block each other.
- Relatively good performance.

Future work

- Exhaustive testing (combining complex intuitionistic and default formulas).
- Optimizations:
  - caching
  - nested implications
  - ...
- Parametric prover on the rules for the underlying logic.